To prove things about det(A) we need a definition...

$$\det(A) = \sum_{\sigma \in S_N} (-)^{\sigma} A_{1\sigma 1} A_{2\sigma 2} A_{3\sigma 3} \cdots A_{N\sigma N}$$

notation for what N is rearranged to

$(-)^{\sigma}$ is the # transpositions in σ even or odd

σ is a permutation of 123...N

Permutations examples: (123) → (312) - 2 transpositions required
(321) - 1
(132) - 1
(1234) → (4123) - 3

A_{12}

Eg: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh$

$\quad 123 \quad 231 \quad 312 \quad 321 \quad 213 \quad 132$

$\underbrace{231}_{A_{12}A_{23}A_{31}}$ $\quad \underbrace{213}_{A_{12}A_{21}A_{33}}$

transpose

To show: $\det(A^T) = \det(A)$

$$= \sum_{\sigma \in S_N} (-)^{\sigma} A_{\sigma 1\, 1} A_{\sigma 2\, 2} A_{\sigma 3\, 3} \cdots A_{\sigma N\, N}$$

$A_{\sigma N\, N}$ is $A^T_{N\,\sigma N}$

Somewhere in this list of row labels there is a 1 (thats what a permutation means). IF $\sigma j = 1$ then $j = \sigma^{-1} 1$ so is $A_{1\,\sigma^{-1}1}$ ... same for 2 etc

$$= \sum_{\sigma \in S_N} (-)^{\sigma} A_{1\,\sigma^{-1}1} A_{2\,\sigma^{-1}2} \cdots A_{N\,\sigma^{-1}N}$$

σ & σ^{-1} have same # swaps $\Rightarrow (-)^{\sigma} = (-)^{\sigma^{-1}}$

$$= \sum_{\sigma \in S_N} (-)^{\sigma^{-1}} A_{1\,\sigma^{-1}1} A_{2\,\sigma^{-1}2} \cdots A_{N\,\sigma^{-1}N}$$

$$= \det(A)$$

To Show: $\det(AB) = \det(A)\det(B)$

$$\det(AB) = \sum_{\sigma \in S_N} (-)^\sigma \left[A_{11}B_{1\sigma 1} + A_{12}B_{2\sigma 1} + \dots + A_{1N}B_{N\sigma 1}\right] \times$$

$$\left[A_{21}B_{1\sigma 2} + A_{22}B_{2\sigma 2} + \dots + A_{2N}B_{N\sigma 2}\right] \times$$

$$\left[A_{31}B_{1\sigma 3} + A_{32}B_{2\sigma 3} + \dots + A_{3N}B_{N\sigma 3}\right] \text{ etc}$$

call this summation index k_1

call this summation index k_2

call this summation index k_3

$$= \sum_{\sigma \in S_N} (-)^\sigma \left[\sum_{k_1} A_{1k_1}B_{k_1\sigma 1}\right]\left[\sum_{k_2} A_{2k_2}B_{k_2\sigma 2}\right] \cdots \left[\sum_{k_N} A_{Nk_N}B_{k_N\sigma N}\right]$$

to expand out, take one term from first [], one from second [] etc

$$= \sum_{k_1, k_2, \dots k_N} \sum_{\sigma \in S_N} (-)^\sigma A_{1k_1}B_{k_1\sigma 1} A_{2k_2}B_{k_2\sigma 2} A_{3k_3}B_{k_3\sigma 3} \cdots$$

$$= \sum_{k_i} A_{1k_1}A_{2k_2}\cdots A_{Nk_N} \underbrace{\left(\sum (-)^\sigma B_{k_1\sigma 1}B_{k_2\sigma 2}B_{k_3\sigma 3}\cdots B_{k_N\sigma N}\right)}$$

if $k_1 = 1$, $k_2 = 2$, $k_3 = 3$ etc this is exactly $\det B$

if any 2 ks are the same this is det of a matrix that has 2 rows the same — that det $= 0$

<u>SO</u> the ks must all be distinct ... a permutation of $123\dots N$ $\rightarrow$ call it P

$$\sum (-)^\sigma B_{p_1\sigma 1}B_{p_2\sigma 2}B_{p_3\sigma 3}\cdots$$

somewhere in this list of row #s there must be a 1 (thats what a permutation is)

if $1 = Pj$ then $j = P^{-1}1$

that term is $B_{1\,\sigma P^{-1}1}$

$$\text{SO} \sim \sum_{\sigma \in S_N} (-)^\sigma B_{1\,\sigma P^{-1}1}B_{2\,\sigma P^{-1}2}B_{3\,\sigma P^{-1}3}\cdots B_{N\,\sigma P^{-1}N}$$

As σ runs over all the elements of S_N σP^{-1} will also cover S_N exactly once. How does $(-)^\sigma$ compare to $(-)^{\sigma P^{-1}}$?

if P^{-1} is even $(-)^\sigma = (-)^{\sigma P^{-1}} = (-)^{\sigma P^{-1}}(-)^{P^{-1}}$

if P^{-1} is odd $(-)^\sigma = -(-)^{\sigma P^{-1}} = (-)^{\sigma P^{-1}}(-)^{P^{-1}}$

$$\text{SO: } (-)^{P^{-1}} \sum (-)^{\sigma P^{-1}} B_{1\,\sigma P^{-1}1}B_{2\,\sigma P^{-1}2}\cdots B_{3\,\sigma P^{-1}3} = (-)^{P^{-1}}\det B$$

So: $\det(AB) = \sum_{p \in S_N} \underset{\uparrow \atop p1}{A_{1k_1}} \underset{\uparrow \atop p2}{A_{2k_2}} \cdots \underset{\uparrow \atop p3}{A_{Nk_N}} (-)^{p'} \det B = \det A \cdot \det B$

Every possible permutation of $k_1 \cdots k_N$ would be in sum of terms

To show: $(AB)^T = B^T A^T$

$$(B^T A^T)_{ij} = \sum B^T_{ik} A^T_{kj} = \sum B_{ki} A_{jk} = (AB)_{ji}$$

$$= (AB)^T{}_{ij}$$

This should remind you of the result for inverses: $(AB)^{-1} = B^{-1} A^{-1}$ as $AB \cdot \underbrace{(B^{-1} A^{-1})}_{\text{so this is } (AB)^{-1}} = 1$

Note: we built up our Euler rotation matrices from 3 simple rotation matrices like: $\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

These simple matrices had the properties:

① $M^{-1} = M^T$

② $\det M = 1$

Applying the above results you should see that the Euler matrices have these same properties.

Remark: The requirement $M^{-1} = M^T$ automatically requires $\det M = \pm 1$ as

$$1 = \det(M M^{-1}) = \det(M M^T) = \det(M) \det(M^T)$$
$$= [\det(M)]^2$$

reflections eg $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ have $\det = -1$

So reflections and/or rotations is the full class $M^{-1} = M^T$

$O(3)$ is this entire class; $SO(3)$ is the special subset that has $\det = +1$ ie pure rotations