Composites of Composites:

CM – add up CM of parts: $\vec{R}_{cm} = \dfrac{M_A \vec{R}_{cm\,A} + M_B \vec{R}_{cm\,B}}{M_A + M_B}$

I – along a single axis just add I of parts

$$\vec{R}_{cm} = \frac{\sum m_\alpha \vec{r}_\alpha}{M} \qquad \vec{V}_{cm} = \frac{\sum m_\alpha \vec{v}_\alpha}{M} \qquad \vec{A}_{cm} = \frac{\sum m_\alpha \vec{a}_\alpha}{M}$$

Because Newton 3 says internal forces cancel: $\sum \vec{F}_\alpha^{ext} = M \vec{A}_{cm}$

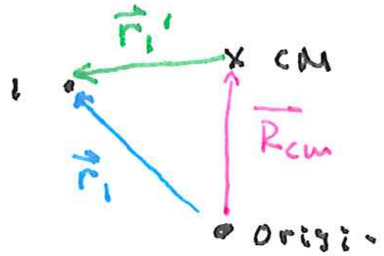
If $\sum \vec{F}_\alpha^{ext} = 0$ then $\vec{A}_{cm} = 0$ so $\vec{V}_{cm} = \text{constant} \Rightarrow \vec{P}_{total} = \text{const}$

If in addition $\vec{V}_{cm} = 0$ then $\vec{R}_{cm} = \text{constant}$

Because Newton 3 & central forces internal torques cancel

$$\sum \vec{\Gamma}_\alpha^{ext} = \frac{d\vec{L}}{dt} \quad \text{where } \vec{L} = \sum \vec{\ell}_\alpha = \sum m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha$$

Express $\vec{L}$ in terms of CM coordinates & CM relative



Note: $\sum m_\alpha \vec{r}_\alpha{}' = 0$; $\sum m_\alpha \vec{v}_\alpha{}' = 0$

$$\vec{r}_\alpha = \vec{R}_{cm} + \vec{r}_\alpha{}'$$
$$\vec{v}_\alpha = \vec{V}_{cm} + \vec{v}_\alpha{}'$$

$$\vec{L} = \sum m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha = \underbrace{M \vec{R}_{cm} \times \vec{V}_{cm}}_{\substack{\text{"orbital"} \\ \text{"OF CM"} \\ \text{depends on origin}}} + \underbrace{\sum m_\alpha \vec{r}_\alpha{}' \times \vec{v}_\alpha{}'}_{\substack{\text{"spin"} \\ \text{"ABOUT CM"} \\ \text{independent of origin}}}$$

Time derivative of spin angular momentum caused by torque calculated about CM –

$$\frac{d\vec{L}}{dt} = \sum \vec{r}_\alpha \times \vec{F}_\alpha^{ext} = M \vec{R}_{cm} \times \vec{A}_{cm} + \frac{d\vec{L}_{spin}}{dt}$$

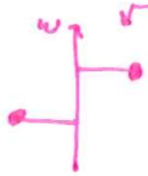
$M\vec{A}_{cm} = \sum \vec{F}_\alpha^{ext}$

so $$\sum (\vec{r}_\alpha - \vec{R}_{cm}) \times \vec{F}_\alpha^{ext} = \frac{d\vec{L}_{spin}}{dt}$$

For rigid bodies $I = \sum m_\alpha r_{\perp\alpha}^2$

(distance from axis to α
Not distance from origin to α

For ~~balanced~~ rigid bodies : $\vec{L}_{spin} = I\vec{\omega}$



not allowed

In this case since CM is on axis $\vec{L} = \vec{L}_{spin}$ is independent of origin but will not be aligned with $\vec{\omega}$

In cases where CM not on axis (ie not calculating $\vec{L}_{spin}$) $\vec{L}_{total}$ will depend on origin so $L = I\omega$ is not at all possible. However if we restrict to origins on axis & only seek $\vec{L}$ in direction of $\vec{\omega}$ (call this z axis) $L_z = I\omega$ & I can be calculated from parallel axis Thm

$$I = I_{cm} + M\underbrace{h}_{\text{distance CM from spin axis}}^{2}$$

Summary: For balanced rigid bodies (the usual case)

$$\vec{L} = M\vec{R}_{cm} \times \vec{V}_{cm} + I\vec{\omega}$$

For objects spinning in an unbalanced way we'll need to make I a 3×3 matrix

Kinetic Energy $T = \frac{1}{2}\sum m_\alpha v_\alpha^2 = \underbrace{\frac{1}{2}MV_{cm}^2}_{\text{KE "OF CM"}} + \underbrace{\frac{1}{2}\sum m_\alpha v_\alpha'^2}_{\text{KE "ABOUT CM" spin}}$

For rotating rigid bodies $T = \frac{1}{2}I\omega^2$

The word "collision" implies no external forces or a collision time so short that the external forces make only negligible changes to momentum — So in both cases momentum is conserved. "Elastic" collisions conserve KE (inelastic collisions do not). "Perfectly inelastic" collisions result in pieces stuck together [so $v_\alpha' = 0$] so the only remaining KE is "OF CM"

The internal PE must depend on coordinate differences $U_{12}(\vec{r_1-r_2})$ where

$$\vec{F}_{12} = -\vec{\nabla}_1 U_{12}$$
$$\vec{F}_{21} = -\vec{\nabla}_2 U_{12}$$

Newton 3 now guaranteed

derivatives wrt location of particle 2

The total internal PE is the sum of all such interacting pairs — $\sum_\alpha \sum_{\beta > \alpha}$ or $\frac{1}{2}\sum\sum_{\alpha \neq \beta}$

Special Case: exactly 2 bodies

Note: $\vec{r_1-r_2} = \vec{r_1'-r_2'}$ $\qquad m_1\vec{r}_1' + m_2\vec{r}_2' = 0$

$\vec{v_1-v_2} = \underbrace{\vec{v_1'-v_2'}}_{\text{recall: CM relative coordinates}}$ $\qquad m_1\vec{v}_1' + m_2\vec{v}_2' = 0$

$$\mu \equiv \frac{m_1 m_2}{m_1+m_2} \quad \text{same as} \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \quad \text{"reduced mass"}$$

$$\vec{v_1-v_2}^{\,2} = \vec{v_1'-v_2'}^{\,2} = v_1'^2 - \vec{v}_1'\cdot\vec{v}_2' + v_2'^2 - \vec{v}_2'\cdot\vec{v}_1'$$
$$= v_1'^2\left(1+\frac{m_1}{m_2}\right) + v_2'^2\left(1+\frac{m_2}{m_1}\right)$$

So $\frac{1}{2}\mu(\vec{v_1-v_2})^2 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2$ ← KE About CM

$$(\vec{r_1-r_2})\times(\vec{v_1-v_2}) = \vec{r_1'}\times\vec{v_1'} - \vec{r}_1'\times\vec{v}_2' + \vec{r}_2'\times\vec{v}_2' - \vec{r}_2'\times\vec{v}_1'$$
$$= (\vec{r}_1'-\vec{r}_2')\times(\vec{v}_1'-\vec{v}_2') = \vec{r}_1'\times\vec{v}_1'\left(1+\frac{m_1}{m_2}\right) + \vec{r}_2'\times\vec{v}_2'\left(1+\frac{m_2}{m_1}\right)$$

So $\mu(\vec{r_1-r_2})\times(\vec{v_1-v_2}) = m_1\vec{r}_1'\times\vec{v}_1' + m_2\vec{r}_2'\times\vec{v}_2'$ ← spin angular momentum

Note: Since T, $\vec{L}$, U_{12} can all be expressed in terms of the relative coordinate $\vec{r_1-r_2}$ that coordinate is generally used for 2 particle systems

Simple Harmonic Oscillator : $F = -kx = m\ddot{x}$

$x = A\sin(\omega t) + B\cos(\omega t) \longleftarrow -\omega^2 x = \ddot{x}$ where $\omega^2 = \frac{k}{m}$

$= \frac{v_0}{\omega} \sin(\omega t) + x_0 \cos(\omega t)$

$-\omega^2 x\dot{x} = \ddot{x}\dot{x}$ integrating factor $\dot{x}$

$-\frac{d}{dt}\left(\frac{\omega^2}{2}x^2\right) = \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2\right)$

$0 = \frac{d}{dt}\underbrace{\left(\frac{1}{2}\dot{x}^2 + \frac{\omega^2}{2}x^2\right)}_{\text{call this constant } \frac{E}{m}}$

$\dot{x}^2 + \omega^2 x^2 = \frac{2E}{m}$

$\dot{x} = \sqrt{\frac{2E}{m} - \omega^2 x^2}$

$\frac{dx}{\sqrt{\frac{2E}{m} - \omega^2 x^2}} = dt \Rightarrow \omega\, dt = \frac{dx}{\sqrt{\frac{2E}{m\omega^2} - x^2}}$

$\omega(t - t_0) = \sin^{-1}\left(\frac{x}{\sqrt{\frac{2E}{m\omega^2}}}\right)\Bigg|_{x_0}^{x}$

Take $t_0 = 0$

call this constant A

$\omega t + \sin^{-1}\left(\frac{x_0}{A}\right) = \sin^{-1}\left(\frac{x}{A}\right)$

$\sin\left(\omega t + \sin^{-1}\left(\frac{x_0}{A}\right)\right) = \frac{x}{A}$

$A\sin\left(\omega t + \sin^{-1}\left(\frac{x_0}{A}\right)\right) = x$

what is relationship between solution $A\sin(\omega t + \phi)$?

$A\sin(\omega t) + B\cos(\omega t)$

$A \leftarrow \frac{v_0}{\omega}$, $B \leftarrow x_0$

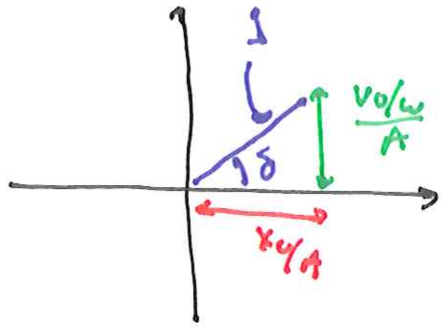
There is a suggestive trig identity:

$\cos\delta \cos\theta + \sin\delta \sin\theta = \cos(\theta - \delta)$

$\cos\delta \leftarrow x_0?$, $\cos\theta \leftarrow \omega t$, $\sin\delta \leftarrow \frac{v_0}{\omega}$, $\sin\theta \leftarrow \omega t$ ← this cannot literally be true as $\cos\delta$ & x_0 have different units.

Define $A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}$

Now x_0/A & $v_0/\omega/A$ can be $\cos\delta$ & $\sin\delta$ as they are both twixt -1 & 1 & sum/square to 1



Construct triangle as shown. Note x_0 and/or v_0 might be negative which just moves triangle out of 1st quadrant

For this triangle $\cos\delta = {}^{v_0}/_A$

$\sin\delta = \dfrac{v_0/\omega}{A}$

$$x = X_0 \cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

$$= \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}\left(\frac{x_0}{\sqrt{}}\cos\omega t + \frac{v_0/\omega}{\sqrt{}}\sin\omega t\right)$$

$$= \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}\left(\cos\delta\cos(\omega t) + \sin\delta\sin\omega t\right)$$

$$= \underbrace{\sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}}\cos(\omega t - \delta)$$

This is $\dfrac{2E}{k} = \dfrac{mv_0^2 + kx_0^2}{k} = \dfrac{v_0^2}{\omega^2} + x_0^2$

UPSHOT: $x = x_0\cos(\omega t) + \dfrac{v_0}{\omega}\sin(\omega t)$

$= A\cos(\omega t - \delta)$ ← given by above triangle

$A = \sqrt{x_0^2 + \dfrac{v_0^2}{\omega^2}} = \sqrt{\dfrac{2E}{k}}$

are equivalent ways to write solution to this diff Eq

Remark: often the result is expressed $\tan\delta = \dfrac{v_0/\omega}{x_0}$

but because $\tan^{-1}$ has range limited to -90 to $+90$

$\tan^{-1}\left(\dfrac{v_0/\omega}{x_0}\right)$ will result in wrong answer if $x_0 < 0$