Multi variable calculus of variations — eg path in polar coordinates: $r(t)$, $\phi(t)$. → generic q_1 & q_2

$$\delta f = \frac{\partial f}{\partial q_1}\delta q_1 + \underbrace{\frac{\partial f}{\partial q_1'}\delta q_1'} + \frac{\partial f}{\partial q_2}\delta q_2 + \underbrace{\frac{\partial f}{\partial q_2'}\delta q_2'}$$

use integration by parts to convert to $-\frac{d}{dx}\left(\frac{\partial f}{\partial q_1'}\right)\delta q_1$

$\hookrightarrow -\frac{d}{dx}\left(\frac{\partial f}{\partial q_2'}\right)\delta q_2'$

$$\Rightarrow \delta I = \int \left\{ \underbrace{\left[\frac{\partial f}{\partial q_1} - \frac{d}{dx}\left(\frac{\partial f}{\partial q_1'}\right)\right]}\delta q_1 + \underbrace{\left[\frac{\partial f}{\partial q_2} - \frac{d}{dx}\left(\frac{\partial f}{\partial q_2'}\right)\right]}\delta q_2 \right\} dx$$

For $\delta I = 0$ both of these terms must be zero

Upshot: the Euler-Lagrange equations apply to each coordinate separately: $\forall_\alpha \quad \frac{\partial f}{\partial q_\alpha} - \frac{d}{dx}\frac{\partial f}{\partial q_\alpha'} = 0$

If f does not explicitly depend on x then:

$$\frac{d}{dx}f = \sum_\alpha \left(\underbrace{\frac{\partial f}{\partial q_\alpha}}_{= \frac{d}{dx}\left(\frac{\partial f}{\partial q_\alpha'}\right)} q_\alpha' + \frac{\partial f}{\partial q_\alpha'} q_\alpha'' \right)$$

$$= \sum_\alpha \frac{d}{dx}\left(\frac{\partial f}{\partial q_\alpha'} q_\alpha'\right)$$

So
$$0 = \frac{d}{dx}\left\{ \underbrace{\sum \frac{\partial f}{\partial q_\alpha'} q_\alpha' - f}_{\text{a constant (like energy)}} \right\}$$

Example: geodesics in 3d space: $x(t)$, $y(t)$, $z(t)$

$$I = \int \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial \dot{x}} = \text{constant} = \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}$$ ← also for y & z

hence $\frac{\dot{x}}{\dot{y}} = \frac{dx}{dy} = \text{const}$ ... geodesics are lines the "energy" constant here is dull:

$$\sum_\alpha \frac{\partial f}{\partial q'_\alpha} q'_\alpha - f = \frac{\dot{x}}{\sqrt{}}\dot{x} + \frac{\dot{y}}{\sqrt{}}\dot{y} + \frac{\dot{z}}{\sqrt{}}\dot{z} - \sqrt{}$$

$$= \frac{(\dot{x}^2+\dot{y}^2+\dot{z}^2) - (\sqrt{})^2}{\sqrt{}} = 0$$

Application to Mechanics: the path followed by particle (ie the path specified by $\vec{F} = m\vec{a}$) minimizes the "Action" $= \int \underbrace{(KE - PE)}_{\equiv \text{ Lagrangian } L} dt$

so... $\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$ for every coordinate

Eg: $L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - U(x,y,z)$

$-\frac{\partial U}{\partial x} = \frac{d}{dt} m\dot{x} \rightarrow$ similar for y & z ie

$-\vec{\nabla} U = m\vec{a}$

Eg: Polar coordinates $- \quad L = \frac{1}{2}m(\dot{r}^2 + (r^2\dot{\phi}^2)) - U(r)$

$$\underbrace{mr\dot{\phi}^2}_{\text{"centrifugal force"}} \underbrace{- \frac{\partial U}{\partial r}}_{\text{expected force}} = \frac{d}{dt}(m\dot{r})$$

$$0 = \frac{d}{dt}(mr^2\dot{\phi}) \longrightarrow \underbrace{mr^2\dot{\phi}}_{\text{angular momentum}} = \text{constant} = \ell$$

Remark: if use angular momentum $mr\dot{\phi}^2 = mr\left(\frac{\ell}{mr^2}\right)^2$

$$= \frac{\ell^2}{mr^3} = -\frac{\partial}{\partial r}\left(\frac{\ell^2}{2mr^2}\right)$$

so $-\frac{\partial}{\partial r}\left(U + \underbrace{\frac{\ell^2}{2mr^2}}_{\text{"centrifugal potential"}}\right) = m\ddot{r}$

An advantage of Lagrange method – no need to worry about "forces of constraint" like Normal force. If, for example, motion is confined to surface of sphere just use coordinates that can describe surface of sphere (and so automatically confine particle to that surface) and use Euler-Lagrange equations for those coordinates and give zero thought about those forces of constraint – the effect of those forces is automatically included.

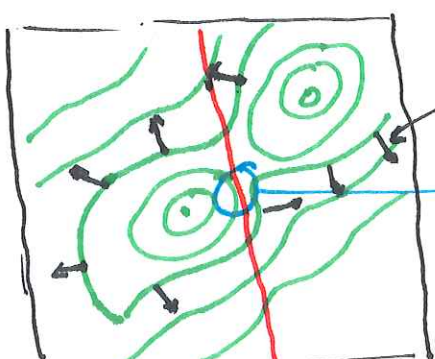
Because "forces of constraint" disappear from view in Lagrange method if you actually want to calculate them you need to do some extra work –

Lagrange Multipliers

Introduction to Lagrange Multipliers. – given a topographical maps (which display lines of constant altitude) and a particular path on that map find the max/min altitude on the path

map: height as function of location: $h = f(x, y)$

path: $g(x, y) = \text{const}$ (eg $x^2 + y^2 = R^2 \leftarrow$ a circle)



↑ path

gradient – points downhill $= -\vec{\nabla} f$

at max $\vec{\nabla} f$ has no component along path – ie $\vec{\nabla} f \parallel \vec{\nabla} g$

so $\vec{\nabla} f = \lambda \vec{\nabla} g$ ← Lagrange Multiplier

or $\vec{\nabla}(f - \lambda g) = 0$