Calculus of Variations: $I = \int f(y(x), y'(x), x)\, dx$

$$\delta I = \int \underbrace{\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right]}\delta y\, dx$$

Euler-Lagrange says this must be zero to "minimize"

Example: Brachistochrome

$$f = \frac{\sqrt{x'^2+1}}{\sqrt{y}}$$

← Here we are integrating wrt y and $x(y)$ is the unknown function

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{y}\sqrt{x'^2+1}} \qquad \text{so} \quad \frac{d}{dy}\frac{\partial f}{\partial x'} = 0$$

$$\frac{\partial f}{\partial x'} = \text{constant}$$

$$\frac{x'^2}{y(x'^2+1)} = \frac{1}{2a}$$

← formula for constant based on knowledge of what makes nice result

$$y\left(1 + \frac{1}{x'^2}\right) = 2a$$

$$\frac{1}{x'^2} = \frac{2a}{y} - 1 = \frac{2a-y}{y}$$

$$x' = \sqrt{\frac{y}{2a-y}}$$

A wildly unlikely trig substitution makes this easier: $y = a(1-\cos\theta)$

$dy = a \sin\theta\, d\theta$

$$x = \int dx = \int \sqrt{\underbrace{\frac{y}{2a-y}}}\, dy$$

$$\frac{a(1-\cos\theta)}{a(1+\cos\theta)} \frac{(1-\cos\theta)}{(1-\cos\theta)} = \frac{(1-\cos\theta)^2}{(1-\cos^2\theta)} = \frac{(1-\cos\theta)^2}{\sin^2\theta}$$

$$= \int \frac{(1-\cos\theta)}{\cancel{\sin\theta}}\, a \cancel{\sin\theta}\, d\theta$$

$$= a(\theta - \sin\theta)$$

Remark: if the end point is $(x,y) = (d,h)$ we must solve the equations: $\frac{d}{h} = \frac{\theta - \sin\theta}{1-\cos\theta}$ ← solve for θ

$h = a(1-\cos\theta)$ ← solve for a

The curve from $(0,0)$ to (d,h) is a part of a cycloid.

Example: soap film area $\quad f = r(x)\sqrt{r'^2+1}$

Note: This f does not depend on x so can use result

$$\frac{\partial f}{\partial r'} r' - f = \text{constant}$$

$$\frac{\partial f}{\partial r'} = \frac{r r'}{\sqrt{r'^2+1}} \quad ; \quad \frac{\partial f}{\partial r'} r' - f = \frac{r r'^2}{\sqrt{r'^2+1}} - r\sqrt{r'^2+1}$$

$$= \frac{r r'^2 - r(r'^2+1)}{\sqrt{r'^2+1}} = \frac{-r}{\sqrt{r'^2+1}}$$

So: $\dfrac{r}{\sqrt{r'^2+1}} = C \leftarrow$ name for this constant

$$\left(\frac{r}{C}\right)^2 = r'^2+1$$

$$\sqrt{\left(\frac{r}{C}\right)^2 - 1} = r'$$

$$dx = \frac{dr}{\sqrt{\left(\frac{r}{C}\right)^2-1}} = \frac{C\,dr}{\sqrt{r^2-C^2}} \Rightarrow x = C\cosh^{-1}\left(\frac{r}{C}\right)$$

$$C\cosh\left(\frac{x}{C}\right) = r$$

"catenary"

Example: geodesics on sphere

$$ds = R\sqrt{d\theta^2 + \sin^2\theta\, d\phi^2}$$

integrand does not depend on ϕ — use integral form

minimize: $\displaystyle\int \sqrt{\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2(\theta)}\, d\phi$

$$\frac{\partial f}{\partial \theta'} = \frac{\theta'}{\sqrt{\theta'^2+\sin^2\theta}} \quad ; \quad \frac{\partial f}{\partial \theta'}\theta' - f = \frac{\theta'^2}{\sqrt{\theta'^2+\sin^2\theta}} - \sqrt{\theta'^2+\sin^2\theta} = \text{constant}$$

$$\Rightarrow \frac{-\sin^2\theta}{\sqrt{\theta'^2+\sin^2\theta}} = \text{Constant}$$

For practice Lets also do the Euler-Lagrange

$$\frac{\partial f}{\partial \theta} = \frac{\sin\theta\cos\theta}{\sqrt{\theta'^2+\sin^2\theta}} \qquad \frac{\partial f}{\partial \theta'} = \frac{\theta'}{\sqrt{\theta'^2+\sin^2\theta}}$$

$$\frac{d}{d\phi}\frac{\partial f}{\partial \theta'} = \frac{\theta''}{\sqrt{\theta'^2+\sin^2\theta}} - \frac{\theta'\left(\theta'\theta'' + \sin\theta\cos\theta\,\theta'\right)}{(\theta'^2+\sin^2\theta)^{3/2}}$$

$$0 = \frac{\partial f}{\partial \theta} - \frac{d}{d\phi}\frac{\partial f}{\partial \theta'} = \frac{\sin\theta\cos\theta}{\sqrt{\theta'^2+\sin^2\theta}} - \left[\frac{\theta''}{\sqrt{\theta'^2+\sin^2\theta}} - \frac{\theta'(\theta'\theta''+\sin\theta\cos\theta\theta')}{(\theta'^2+\sin^2\theta)^{3/2}}\right]$$

$$\to \frac{\theta''(\theta'^2+\sin^2\theta) - \theta'^2\theta''}{(\theta'^2+\sin^2\theta)^{3/2}} = \frac{\theta''\sin^2\theta}{(\theta'^2+\sin^2\theta)^{3/2}}$$

Mult both sides by $(\theta'^2+\sin^2\theta)^{3/2}$

$$0 = \sin\theta\cos\theta(\theta'^2+\sin^2\theta) - \theta''\sin^2\theta + \sin\theta\cos\theta\,\theta'^2$$

Divide both sides by $\sin\theta$

$$0 = -\sin\theta\,\theta'' + 2\cos\theta\,\theta'^2 + \sin^2\theta\cos\theta$$

With lots of work one can show solution is

$$\cot\theta = A\sin(\phi+\phi_0)$$ [Note: 2 adjust constants]

But there's no reason we need to spend time showing this result, if you really want to verify this works use the integral formula.