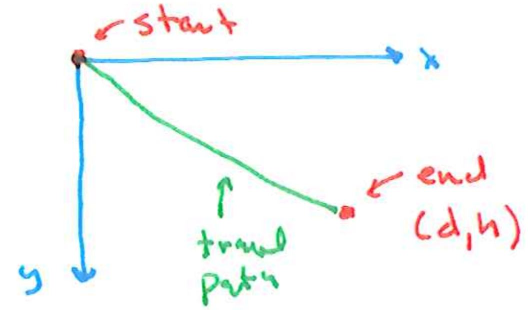
Calculus of Variations $\underbrace{\text{finds}}$ the function that $\underbrace{\text{minimizes}}$ an integral

↳ ie gives use the diff eq for the function
we still must solve the diff eq

or maxs
or inflection

we now define "minimize" to mean any of these options

Example: Brachistochrone : Given a particle must travel between two fixed points, what curve minimizes the time.

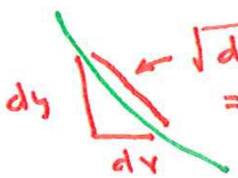


For this falling object the speed can be determined by conservation of energy

$$E = \tfrac{1}{2}mv^2 + mgh$$

starts at y=0
with v=0
so E=0

$h = -y$ with this downward coordinate system



$$\sqrt{dy^2+dx^2} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$\Rightarrow v = \sqrt{2gy}$$

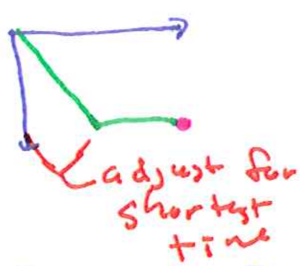
$$\text{time} = \text{sum of } \frac{\text{distance step}}{\text{speed}} = \int \frac{\sqrt{1+\frac{dx}{dy}^2}\,dy}{\sqrt{2gy}}$$

Some examples of curve & resulting time

→ straight line $\sqrt{1+\left(\frac{d}{h}\right)^2}\sqrt{\frac{2h}{g}} \rightarrow \frac{2}{\sqrt{g}}$ in case $d=h=1$

→ straight down & over $\sqrt{\frac{2h}{g}} + \frac{d}{\sqrt{2gh}} \rightarrow \left(\sqrt{2}+\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{g}}$

(slightly more)

→ 

→ $\frac{1.93}{\sqrt{g}}$ (better)

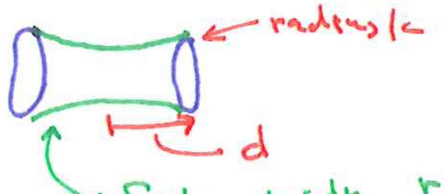
→ 

3 line segments with slopes designed to minimize time

→ $\frac{1.87}{\sqrt{g}}$ (better still) we could continue with this "guess & check" approach but its hard to see how we could even prove our path was "shortest"

Example: minimize the energy (ie area as surface tension is $\frac{\text{energy}}{\text{area}}$) of a soap film supported between 2 hoops

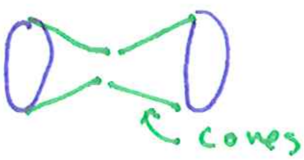
Curve in to get smaller circumference at the cost of slant distance



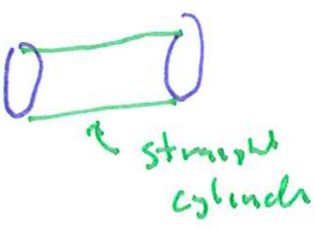
Film with radius $r(x)$. Note $r(\pm d) = R$ as film must connect with hoops

$$\text{Area} = \int_{-d}^{d} \underbrace{2\pi r(x)}_{\text{circumference}} \underbrace{\sqrt{r'(x)^2 + 1}}_{\text{slant distance}} \, dx$$

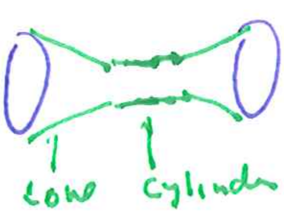
Guess-Check — (case $d = \frac{1}{4}$, $R = 1$)



Area = 3.117



Area = π



Area = 3.112

Guess-check of various functions is never going to show that we've found the best function
(But the guess should get 'close' to the correct answer)

Summary: Brachistochrone: minimize $\int \frac{\sqrt{1+x'^2}}{\sqrt{y}}\,dy$

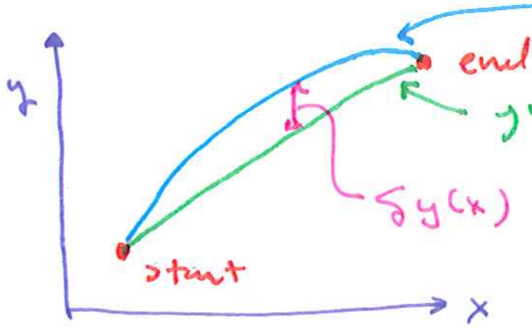
Soap Film: minimize $\int r(x)\sqrt{1+r'^2}\,dx$

In general: minimize $\int f(y(x), y'(x), x)\,dx$

→ In first example $x(y) \to y(x)$ & integrate $dy \to dx$

In second example $r(x) \to y(x)$

modification: new $y(x) = y(x) + \delta y(x)$



guessed function $y(x)$

Note: $\delta y|_{end} = 0$

$$f(y+\delta y,\ y'+\delta y',\ x) \approx f(y, y', x) + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'$$

$$I = \int f(y, y', x)\,dx$$

$$I + \delta I = \int f(y+\delta y,\ y'+\delta y',\ x)$$

substitute

$$= I + \underbrace{\int \left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'\right]dx}_{}$$

This shows how much integral changes when we modify our function

$\equiv \delta I$

Now: $$\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\delta y\right] = \frac{\partial f}{\partial y'}\delta y' + \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\delta y$$

$$\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\delta y\right] - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\delta y = \frac{\partial f}{\partial y'}\delta y'$$

$$\delta J = \int \Big[\frac{\partial f}{\partial y}\,\delta y + \underbrace{\frac{d}{dx}\Big(\frac{\partial f}{\partial y'}\,\delta y\Big)} - \frac{d}{dx}\Big(\frac{\partial f}{\partial y'}\Big)\,\delta y \Big]\, dx$$

$$\frac{\partial f}{\partial y}\,\delta y\Big|_{ends} = 0 \text{ as } \delta y\Big|_{ends} = 0$$

$$= \int \underbrace{\Big[\frac{\partial f}{\partial y} - \frac{d}{dx}\Big(\frac{\partial f}{\partial y'}\Big) \Big]}\, \delta y \, dx$$

if this quantity is positive some place in the interval included in the range of integration, we can choose $\delta y < 0$ in that spot $\Rightarrow \delta J < 0$ ie can do better

if this quantity is negative in some region included in range of integration we can select $\delta y > 0$ there $\Rightarrow \delta J < 0$ ie can do better.

Upshot: for the best (minimizing) curve that quantity must be zero.

Eg $\quad f = \dfrac{\sqrt{1+x'^2}}{\sqrt{y}} \rightarrow 0 - \dfrac{d}{dy}\left(\dfrac{x'}{\sqrt{1+x'^2}} \dfrac{1}{\sqrt{y}} \right) = 0$

there is no $x(y)$ function in this f

$f = r(x)\sqrt{1+r'^2} \longrightarrow \sqrt{1+r'^2} - \dfrac{d}{dx}\left(r \dfrac{r'}{\sqrt{1+r'^2}} \right) = 0$

The last example has a special property which allows us to show a constant of diff eq equivalent to energy

if $\dfrac{\partial f}{\partial x} = 0$ then $\dfrac{d}{dx} f = \underbrace{\dfrac{\partial f}{\partial y}}_{= \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)} y' + \dfrac{\partial f}{\partial y'} y''$

$$= \frac{d}{dx}\Big(\frac{\partial f}{\partial y'}\, y' \Big)$$

so $\dfrac{d}{dx}\Big(f - \dfrac{\partial f}{\partial y'} y' \Big) = 0$

so $\quad f - \dfrac{\partial f}{\partial y'} y' = \text{constant}$