Green's Functions: The solution to $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$

is $x(t) = \int_{-\infty}^{t} f(t')\, \underbrace{G(t,t')}_{\text{Green's Function}}\, dt'$

Greens Function is itself the solution to

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \underbrace{\delta(t-t')}_{\text{Dirac Delta function}}$$

The Dirac Delta function represents an impulsive force – a hammer blow.

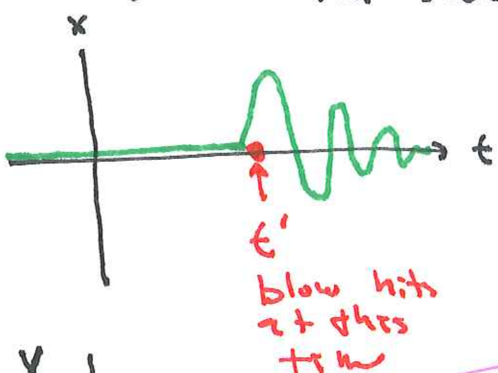
In 191 "impulse" was Force × time or better

$\int F(t)dt$ ← the area under Force vs time plot.

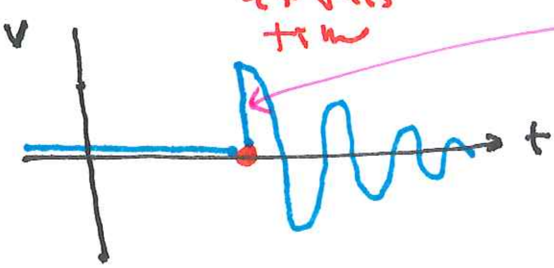
Because $F = \frac{dp}{dt}$, $\int F\,dt = \Delta p$ ← a sudden change in p

In the problem at hand we've divided out the mass so "f(t)" is really $\frac{\text{Force}}{\text{mass}}$ = acceleration so we really have $\int F(t)\,dt = \Delta(\text{velocity})$

Consider a hammer blow to our SHO at t':

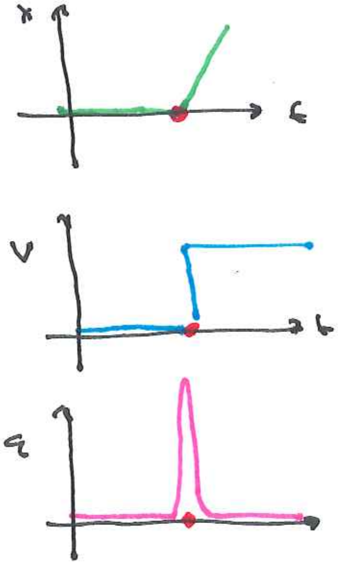


Greens function is basically this hammer blow solution.



sudden discontinuous change in velocity
we will want this to be one so we can easily generate larger/smaller blows by simple multiplication

expand view near hammer blow time t'



EE ~1900

Heaviside Function

$\theta(t-t')$

UnitStep in Mathematica

Dirac Delta function $\delta(t-t') = \frac{d}{dt}\theta(t-t')$

PAD ~1930

Mathematicians may tremble at the thought of taking the derivative of a discontinuity but physicists & EE are made of sterner stuff

Seeking unit step damped SHO solution that is the Greens function.

at $t=t'$ starts at $x=0$

$$G(t,t') = \frac{1}{\omega_1} e^{-\beta(t-t')} \sin(\omega_1(t-t'))$$

For $t > t'$ (is zero for $t < t'$)

free osc freq

needed to give $x'(0) = 1$ ← unit change in velocity

Example: Constant force → result stretched spring $x = \frac{f_0}{\omega_0^2}$

constant!

$$x(t) = \int_{-\infty \to \infty}^{t \to 0} f_0 \frac{1}{\omega_1} e^{-\beta\underbrace{(t-t')}_{u}} \sin(\omega_1\underbrace{(t-t')}_{u})\, \underbrace{dt'}_{-du}$$

$$= \int_0^\infty \frac{f_0}{\omega_1} e^{-\beta u} \sin(\omega_1 u)\, du = \frac{f_0}{\omega_1} \mathrm{Im}\left[\int_0^\infty \underbrace{e^{-\beta u} e^{i\omega_1 u}}_{e^{-(\beta - i\omega_1)u}} du\right]$$

can already see does not depend on t

$$= \frac{f_0}{\omega_1} \mathrm{Im}\left[\underbrace{\frac{1}{\beta - i\omega_1} \times \frac{\beta + i\omega_1}{\beta + i\omega_1}}\right] \qquad \frac{\beta + i\omega_1}{\beta^2 + \omega_1^2} = \frac{\beta + i\omega_1}{\omega_0^2}$$

$$= \frac{f_0}{\omega_1} \frac{\omega_1}{\omega_0^2} = \frac{f_0}{\omega_0^2} \checkmark$$

Example: $f(t) = \cos(\omega t) = \mathrm{Re}\left[e^{i\omega t}\right]$

$$x(t) = \int_{-\infty \to \infty}^{t \to 0} \cos(\omega t') \frac{1}{\omega_1} e^{-\beta(t-t')} \sin(\omega_1 \underbrace{(t-t')}_{u}) \underbrace{dt'}_{-du}$$

$$\cos(\omega t') \to \mathrm{Re}\, e^{i\omega t'} = \mathrm{Re}\left[e^{i\omega(t'-t+t)}\right]$$

$$= \mathrm{Re}\left[\int_0^\infty e^{i\omega(-u+t)} \frac{1}{\omega_1} e^{-\beta u} \underbrace{\sin(\omega_1 u)}_{\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}} du\right]$$

$$= \frac{1}{\omega_1} \mathrm{Re}\left[e^{i\omega t} \int_0^\infty e^{-i\omega u} e^{-\beta u} \left(\frac{e^{i\omega_1 u} - e^{-i\omega_1 u}}{2i}\right) du\right]$$

$$= \frac{1}{\omega_1} \mathrm{Re}\left[\frac{e^{i\omega t}}{2i} \int_0^\infty \left(e^{-(\beta + i\omega - i\omega_1)u} - e^{-(\beta + i\omega + i\omega_1)u}\right) du\right]$$

$$= \frac{1}{\omega_1} \mathrm{Re}\left[\frac{e^{i\omega t}}{2i} \left(\frac{1}{\beta + i(\omega-\omega_1)} - \frac{1}{\beta + i(\omega+\omega_1)}\right)\right]$$

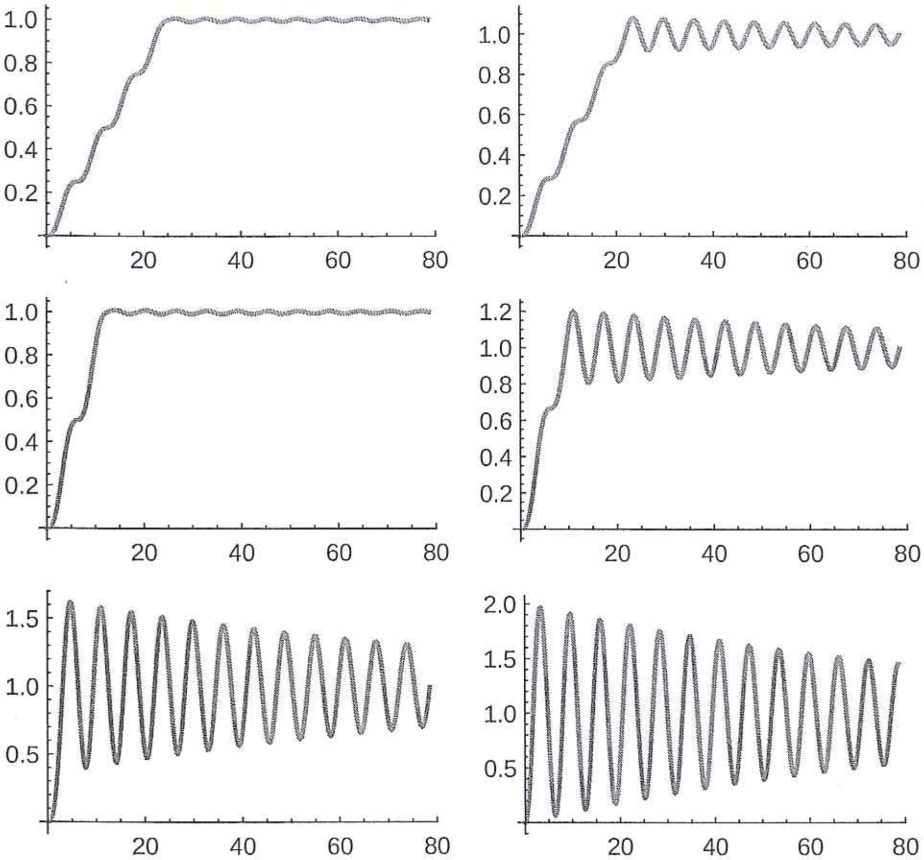
$$= \frac{1}{\omega_1} \mathrm{Re}\left[\frac{e^{i\omega t}}{2i} \frac{\overbrace{\beta + i(\omega+\omega_1) - [\beta + i(\omega-\omega_1)]}^{2i\omega_1}}{\underbrace{(\beta + i(\omega-\omega_1))(\beta + i(\omega+\omega_1))}_{\beta^2 + 2\beta i\omega - (\omega^2-\omega_1^2)}}\right]$$

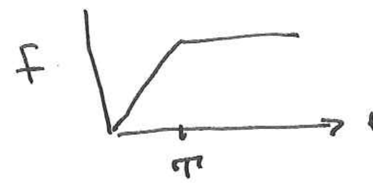
$$\omega_1^2 + \beta^2 = \omega_0^2$$

$$= \frac{1}{\omega_1} \mathrm{Re}\left[\frac{e^{i\omega t}}{2i} \frac{2i\omega_1}{\omega_0^2 - \omega^2 + 2i\beta\omega}\right]$$

$$= \mathrm{Re}\left[\frac{1}{\omega_0^2 - \omega^2 + 2i\beta\omega} e^{i\omega t}\right] \checkmark$$

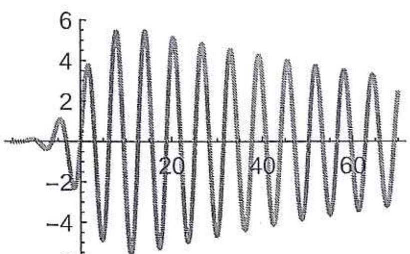
Numerical example
$\omega_0 = 1$, $b = .01$





f rises linearly reaches 1 at $t = T$.

If linear ramp is "slow" expect little oscillation



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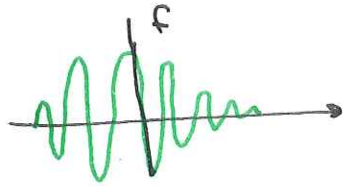

f oscillates at resonant freq but turns on & off over time $\approx \frac{1}{\sqrt{a}}$

$a = .02$


```
w1=Sqrt[1-b^2]
begin=Integrate[tp/T Exp[-b (t-tp)]Sin[w1(t-tp)],{tp,0,t}]/w1
first=Integrate[tp/T Exp[-b (t-tp)]Sin[w1(t-tp)],{tp,0,T}]/w1
end=Integrate[ Exp[-b (t-tp)]Sin[w1(t-tp)],{tp,T,t}]/w1
x[t_]=If[t<T,Evaluate[begin],Evaluate[first+end]]
b=.01
T=8 Pi
sloww=Plot[x[t],{t,0,25 Pi},PlotRange->All]
T=7 Pi
sloww0=Plot[x[t],{t,0,25 Pi},PlotRange->All]
T=4 Pi
slow=Plot[x[t],{t,0,25 Pi},PlotRange->All]
T=3Pi
slow0=Plot[x[t],{t,0,25 Pi},PlotRange->All]
T=Pi
fast=Plot[x[t],{t,0,25 Pi},PlotRange->All]
T=Pi/10
faster=Plot[x[t],{t,0,25 Pi},PlotRange->All]
GraphicsGrid[{{sloww,sloww0},{slow,slow0},{fast,faster}}]

a=.02
f[t_]=Integrate[Exp[-a tp^2]Cos[tp]Exp[-b(t-tp)]Sin[w1(t-tp)]/w1,{tp,-Infinity,t}]
gauss=Plot[Re[f[t]],{t,-15,70}]
GraphicsGrid[{{sloww,sloww0},{slow,slow0},{fast,faster},{gauss}}]
```

Why is $x(t) = \int_{-\infty}^{t} f(t')\, G(t,t')\, dt'$ a solution?

(A) This integral is superposition of lots of little hammer blows ($f(t')\Delta t'$) in the past ($t' < t$)

$$\int_{-\infty}^{t} f(t')\, G\, dt = \sum_{\text{past}} f(t')\Delta t\, G(t,t')$$ ← superposition of past hammer blows

(B) Notation $D\,x(t) \equiv \ddot{x} + 2\beta\dot{x} + \omega_0^2 x$

$G(t,t')$ is solution where hammer hits at $t=t'$

ie $D\,G(t,t') = \delta(t-t')$

Both integrals almost always zero

Now in general $\int g(x)\,\delta(x-x_0)\,dx = \int g(x_0)\,\delta(x-x_0)\,dx$

$$= g(x_0)\underbrace{\int \delta(x-x_0)\,dx}_{\text{area under } \delta \text{ is } 1} = g(x_0)$$

So $D\int^{t} f(t')\,G(t,t')\,dt' = \int^{t} f(t')\,D\,G(t,t')\,dt'$

D operates on t Not t'

$$= \int^{t} f(t')\,\delta(t-t')\,dt' = f(t)$$