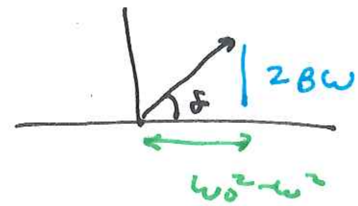
Problem: Driven, damped SHO where the driving force has a complex wave shape (ie is Not pure sinusoid)

Solution: write the driving force as a sum of sin & cos at harmonics of driving freq. We know the particular solution for a sin or cos at any freq:

$$x = A(\omega)\begin{matrix}\sin(\omega t-\delta)\\ \cos(\omega t-\delta)\end{matrix} \qquad A = \frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+(2\beta\omega)^2}}$$

$\delta =$




By linearity the particular solution for the complex driving force is just the sum of those terms:

$$x = \sum A(k\omega)\, f_{ck} \cos(k\omega t - \delta(k\omega)) + \sum A(k\omega)\, f_{sk} \sin(k\omega t - \delta(k\omega))$$

where $f(t) = \sum f_{ck} \cos(k\omega t) + f_{sk} \sin(k\omega t)$

Main point: if we can find these coefficients we can find $x(t)$ just by doing the sum.

Comment: if one of the $k\omega$ is near the resonance frequency $\omega_0$ that one term will be much larger than the others — you will see approximately sinusoidal response in $x$ at nearly $\omega_0$.

IE. if a harmonic of the driving frequency matches the resonance frequency you'll get a big amplitude $x$ motion.

Problem: Given a periodic ~~For~~ driving function (period T) how can you calculate the "expansion coefficients"

$$f(t) = \sum \underbrace{f_{ck}}_{} \cos(k\omega t) + \underbrace{f_{sk}}_{} \sin(k\omega t)$$

expansion coefficients

Closely related & easier problem: given periodic function $f(t)$ find complex expansion coefficients.

$$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{ik\omega t}$$

or any full period of the driving force eg $\int_0^T$

Answer: $f_k = \frac{1}{T} \int_{-T/2}^{T/2} e^{-ik\omega t} f(t)\, dt$

Note: From this result its not hard to show the book's result for sin/cos expansion:

$$f_{ck} = \frac{2}{T} \int_{-T/2}^{T/2} \cos(k\omega t) f(t)\, dt$$

$$f_{sk} = \frac{2}{T} \int_{-T/2}^{T/2} \sin(k\omega t) f(t)\, dt$$

I hope you can see the relationship between these results

Note: Mathematica has a function to find these expansion coefs [both Trig & Complex]

Something like a proof.

1) Think of functions as vectors where components are denoted not by integers like 1, 2, 3 but rather $x$

eg if $\vec{A} = (7, 6, 9)$ $\quad A_2 = 6 \quad A(2) = 6$

$f = t^2 \quad f_{\sqrt{2}} = 2 \quad f(\sqrt{2}) = 2$

clearly this is an infinite dimensional vector space.

2) seek orthogonal unit vectors in this space ie things like $\hat{i}, \hat{j}, \hat{k}$ that we use to express 3d vectors in.

3) Note: $\vec{A} \cdot \vec{B} = \sum A_i B_i$ so dot product of 2 functions $f \cdot g = \int f(x)\, g(x)\, dx$

Since we require $f \cdot f \geq 0$ & we have complex vectors define $f \cdot g = \int f^*(x)\, g(x)\, dx$ ← complex conjugate

4) The functions $e_k(x) = e^{ik\omega x}$ are orthogonal

$$e_k \cdot e_l = \int_0^T e^{-ik\omega x} e^{il\omega x}\, dx = \int_0^T e^{i(l-k)\omega x}\, dx$$

$$= \frac{1}{i(l-k)\omega} e^{i(l-k)\omega} \Big|_0^T = 0$$

5) Note to make them unit vectors : $\frac{1}{\sqrt{T}} e_k(x) \equiv \hat{e}_k(x)$

6) Now as usual if $\vec{A} = (6, 3, 1)$, then $A_2 = \vec{A} \cdot \hat{j} = 3$

so if $f(t) = \sum f_k \hat{e}_k(t)$ then

$$\hat{e}_l \cdot f(t) = \int_0^T \left(\frac{1}{\sqrt{T}} e^{il\omega t}\right)^* f(t)\, dt = f_l$$

This is called "Fourier's Trick"