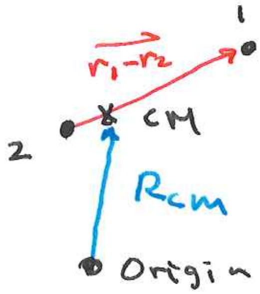
2 body systems: $\vec{r}_1$ & $\vec{r}_2$ (assume $m_2 > m_1$)
m_1 m_2

$U(\vec{r_1 - r_2}) \rightarrow$ ie no external force - just depends on relative distance



$$T = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} \mu v^2$$

$$\vec{V}_{cm} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{M}$$

$v = \dot{\vec{r_1 - r_2}}$

Since no external forces $V_{cm} = $ constant

$$L = \frac{1}{2} \mu v^2 - U(\vec{r})$$

$$\text{angular momentum} = \vec{L} = \underbrace{M \vec{R_{cm} \times V_{cm}}}_{\text{orbital or "of CM"}} + \underbrace{\overbrace{\mu \vec{r} \times \vec{v}}^{\text{call } \vec{\ell}}}_{\text{spin or "about CM"}}$$

This will be constant as
$\vec{R}_{cm} = \vec{R}_{cm}(0) + \vec{V}_{cm} t$
$\vec{V}_{cm} \times \vec{V}_{cm} = 0$

For central forces this must be a constant as zero torque about CM

Remark: if $\vec{A \times B} = \vec{C}$ then $\vec{A}$ & $\vec{B}$ must both be $\perp$ to $\vec{C}$ so $\vec{A}$ & $\vec{B}$ "live" in the plane $\perp$ to $\vec{C}$

Forces directed between 1 & 2 so $\vec{r \times F} = 0$

Here $\mu \vec{r} \times \vec{v} = \vec{\ell} = $ constant so $\vec{r}$ & $\vec{v}$ live in the plane $\perp$ to $\vec{\ell} \rightarrow$ use polar coordinates

$$\left.\begin{aligned} \vec{r} &= r\hat{r} \\ \vec{v} &= \dot{r}\hat{r} + (r\dot{\phi})\hat{\phi} \end{aligned}\right\} \mu \vec{r \times v} = \mu \underbrace{(r^2 \dot{\phi})}_{\text{must be constant} = \ell} \underbrace{\hat{r} \times \hat{\phi}}_{\text{call this constant vector } \hat{z}}$$

$$v^2 = (\dot{r}^2 + (r\dot{\phi})^2)$$

$$L = \frac{1}{2}\mu(\dot{r}^2 + (r\dot{\phi})^2) - U(r)$$

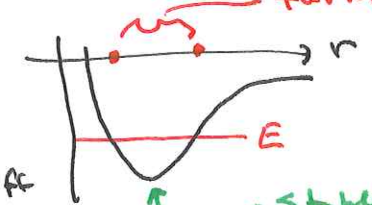
$$\boxed{r} \quad \mu \ddot{r} = \mu r \dot{\phi}^2 - \frac{\partial U}{\partial r} = \mu r \left(\frac{\ell}{\mu r^2}\right)^2 - \frac{\partial U}{\partial r} = \frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r} = \frac{-\partial}{\partial r}\left(\frac{\ell^2}{2\mu r^2} + U\right)$$

$$\boxed{\phi} \quad \frac{d}{dt} \mu r^2 \dot{\phi} = 0 \Rightarrow \mu r^2 \dot{\phi} = \text{constant} = \ell$$

$$\mu\ddot{r} = -\frac{\partial}{\partial r}\Big(\underbrace{\frac{\ell^2}{2\mu r^2} + u(r)}_{u_{eff}}\Big) \rightarrow \frac{(\mu r^2\dot{\phi})^2}{2\mu r^2} = \frac{1}{2}\mu\,\underbrace{(r\dot{\phi})^2}_{v_\perp^2} = \text{transverse KE}$$

$$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) = -\frac{d}{dt}u_{eff} \Rightarrow \frac{1}{2}\mu\dot{r}^2 + u_{eff} = \text{Const} = KE + PE = E$$

Generic Case:



stable equilibrium where $\overbrace{\frac{\ell^2}{\mu r^3} - \frac{\partial u}{\partial r}}^{f(r)} = 0$

For small amplitude oscillation freq seek $f'(r_0)$

$$f'(r) = \frac{-3\ell^2}{\mu r^4} - \frac{\partial^2 u}{\partial r^2}$$

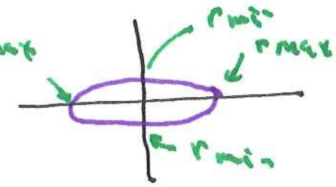
Eg: $u = \frac{1}{2}kr^2 \longrightarrow f(r) = \frac{\ell^2}{\mu r^3} - kr = 0$ iff $\frac{\ell^2}{\mu k} = r^4$

$$f'(r) = \frac{-3\ell^2}{\mu r^4} - k \quad @ \quad r^4 = \frac{\ell^2}{\mu k}$$

$$= -4k$$

so $\mu\Delta\ddot{r} = -4k\Delta r \Rightarrow \omega = \sqrt{\frac{4k}{m}} = 2\sqrt{\frac{k}{m}}$

2 cyles in r per cycle in ϕ



Eg $u = \frac{-\alpha}{r} \longrightarrow f(r) = \frac{\ell^2}{\mu r^3} - \frac{\alpha}{r^2}$

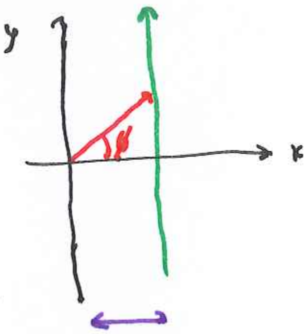
$$= \frac{\ell^2 - \mu\alpha r}{\mu r^3} = 0 \text{ if } r = \frac{\ell^2}{\mu\alpha}$$

$$f'(r_0) = \frac{-\mu\alpha}{\mu r_0^3} = \frac{-\alpha}{r_0^3} \Rightarrow \omega = \sqrt{\frac{\alpha}{\mu r_0^3}}$$

ck: $\underbrace{\mu\omega^2 r}_{\text{Centripetal acceleration}} = \frac{\alpha}{r^2} \Rightarrow \omega = \sqrt{\frac{\alpha}{\mu r^3}}$

$\frac{1}{r^2}$ Force

Eg free particle $U = 0$



impact parameter b

$$\left.\begin{aligned} y &= v_0 t \\ x &= b \end{aligned}\right\} \tan\phi = \frac{v_0 t}{b} \qquad \text{Note } 1 + \tan^2 = \frac{1}{\cos^2}$$

$$r = \frac{b}{\cos\phi} = b\sqrt{1 + \left(\frac{v_0 t}{b}\right)^2} = \sqrt{b^2 + (v_0 t)^2}$$

$$\frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} = \frac{1}{2}\mu v_0^2$$

$$\dot{r}^2 = v_0^2 - \frac{\ell^2}{\mu^2 r^2} = b^2$$

$$= v_0^2\left(1 - \frac{\ell^2}{v_0^2 \mu^2 r^2}\right)$$

$$\ell = \mu r v_0 \sin(90 - \phi)$$

$$= \mu r v_0 \cos\phi = \mu b v_0$$

$$\dot{r} = v_0 \sqrt{1 - \frac{b^2}{r^2}}$$

$$u = r^2$$

$$\frac{\frac{1}{2} du}{\sqrt{u - b^2}} = \frac{r\,dr}{\sqrt{r^2 - b^2}} = \frac{dr}{\sqrt{1 - \frac{b^2}{r^2}}} = v_0\, dt$$

$$(u - b^2)^{1/2}\Big|_{b^2}^{r^2} = (r^2 - b^2)^{1/2} = v_0 t$$

$$r = \sqrt{(v_0 t)^2 + b^2}$$

Now seek $\phi(t)$:

$$\ell = \mu r^2 \dot{\phi} = \mu\left((v_0 t)^2 + b^2\right)\dot{\phi}$$

$$\frac{b}{v_0} = \left(t^2 + \frac{b^2}{v_0^2}\right)\dot{\phi}$$

$$\frac{\frac{b}{v_0}\, dt}{(t^2 + b^2/v_0^2)} = d\phi$$

$$\Downarrow$$

$$\frac{b}{v_0}\,\frac{1}{\frac{b}{v_0}}\tan^{-1}\left(\frac{t}{b/v_0}\right) = \phi$$

$$\frac{t}{b/v_0} = \tan\phi$$

$$\frac{v_0 t}{b} = \tan\phi$$

Instead of having the parametric eqs $r(t)$ & $\phi(t)$ some times seek the "orbit" re geometry w/o time: $r(\phi)$

Orbit eqn: $\mu r^2 \dot{\phi} = \ell$

$$\frac{dr}{d\phi} = \frac{1}{\dot{\phi}}\frac{dr}{dt} = \frac{\mu r^2}{\ell}\frac{dr}{dt}$$

$$u = \frac{1}{r} \qquad \frac{du}{d\phi} = \frac{-1}{r^2}\frac{dr}{d\phi} = -\frac{\mu}{\ell}\frac{dr}{dt}$$

So conservation of energy: $\frac{1}{2}\mu \underset{\uparrow \atop \frac{\ell}{\mu}\frac{du}{d\phi}}{\dot{r}}^2 + \frac{\ell^2}{2\mu \underset{\uparrow \atop u^2}{r^2}} + U(r) = E$

For free particle: $\frac{1}{2}\frac{\ell^2}{\mu}\left(\frac{du}{d\phi}\right)^2 + \frac{\ell^2}{2\mu}u^2 + 0 = \frac{1}{2}\mu v_0^2$

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{\mu^2 v_0^2}{\ell^2} = \frac{1}{b^2}$$

$$\frac{du}{\sqrt{\frac{1}{b^2} - u^2}} = d\phi$$

$$\cos^{-1}\left(\frac{u}{1/b}\right) = \phi$$

$$u = \frac{1}{b}\cos\phi$$

$$\frac{b}{r} = \cos\phi$$