In 1879 Josiah Willard Gibbs published[1] an alternative formulation for Newtonian mechanics. Just as Lagrange's formulation produces equations identical to those from $\mathbf{F} = m\mathbf{a}$ but deals more easily with constraints, so Gibbs' formulation produces the same equations as Newton's but deals more easily with non-holonomic constraints. Gibb's formulation looks quite simple. Form what looks like total kinetic energy of all N particles but use accelerations not velocities:

$$\mathfrak{G} = \sum_{i=1}^{N} \frac{1}{2}\, m_i a_i^2 \tag{1}$$

Express the result in terms of generalize coordinate accelerations ($\ddot{q}_r$ for $r \in \{1 \dots k\}$). Find the work done for displacements in those generalized coordinates:

$$dW = \sum_{r=1}^{k} Q_r dq_r \tag{2}$$

(The Q_r are called generalized forces.) Then

$$\frac{\partial \mathfrak{G}}{\partial \ddot{q}_r} = Q_r \tag{3}$$

Note that contributions to $\mathfrak{G}$ that do not depend on the $\ddot{q}_r$ will play no role in the equations of motion and will be dropped often without comment.

Examples

Example 1: A particle responding to a potential $V(x, y, z)$:

$$\begin{aligned}
\mathfrak{G} &= \frac{1}{2}\, m(\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2) && (4)\\
\mathbf{Q} &= -\boldsymbol{\nabla} V && (5)\\
m\mathbf{a} &= -\boldsymbol{\nabla} V && (6)
\end{aligned}$$

Example 2a: A particle responding to a central potential $V(r)$ (polar coordinates):

$$\begin{aligned}
\mathfrak{G} &= \frac{1}{2}\, m(\ddot{x}^2 + \ddot{y}^2) = \frac{1}{2}\, m\left((\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2\right) && (7)\\
&= \frac{1}{2}\, m\left(\ddot{r}^2 - 2r\ddot{r}\dot{\theta} + r^2\ddot{\theta}^2 + 4r\dot{r}\dot{\theta}\ddot{\theta}\right) && (8)\\
m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) &= 0 = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = \frac{dL}{dt} && (9)\\
m\left(\ddot{r} - r\dot{\theta}^2\right) &= -\partial_r V && (10)\\
m\ddot{r} &= -\partial_r V + \frac{L^2}{mr^3} && (11)
\end{aligned}$$

[1] Gibbs, JW (1879). "On the Fundamental Formulae of Dynamics.". American Journal of Mathematics. 2: 49–64. At the time Gibb's work was mostly ignored; Paul Émile Appell independently rediscovered this formulation in 1900.

Example 2b: Instead of using the holonomic coordinate θ we can use the non-holonomic coordinate $dq = x\ dy - y\ dx$. (dq is twice the area swept by $\mathbf{r}$ as the particle goes from (x, y) to $(x + dx, y + dy)$ and is therefore closely related to Kepler's second law and angular momentum— note: $m\dot{q} = L$, but clearly q depends on history not current configuration.)

$$\begin{aligned}
\mathfrak{G} &= \frac{1}{2}\, m(\ddot{x}^2 + \ddot{y}^2) = \frac{1}{2}\, m \left\{ \left(\ddot{r} - \frac{\dot{q}^2}{r^3} \right)^2 + \frac{\ddot{q}^2}{r^2} \right\} && (12) \\
&= \frac{1}{2}\, m \left(\ddot{r}^2 - \frac{2\ddot{r}\dot{q}^2}{r^3} + \frac{\ddot{q}^2}{r^2} \right) && (13) \\
\frac{m}{r^2}\, \ddot{q} &= 0 \qquad\qquad \text{i.e., } \dot{q}\text{=constant=}L/m && (14) \\
m \left(\ddot{r} - \frac{\dot{q}^2}{r^3} \right) &= -\partial_r V && (15) \\
m\ddot{r} &= -\partial_r V + \frac{L^2}{mr^3} && (16)
\end{aligned}$$

Note that while the Lagrangian

$$L = \frac{1}{2}\, m \left(\dot{r}^2 + \frac{\dot{q}^2}{r^2} \right) - V(r) \tag{17}$$

is KE–PE, it produces incorrect equations of motion:

$$\begin{aligned}
\frac{m\dot{q}}{r^2} &= \text{constant} && (18) \\
m\ddot{r} &= -\partial_r V - \frac{m\dot{q}^2}{r^3} && (19)
\end{aligned}$$

because q is non-holonomic.

Example 3: Pseudo-forces on a rotating plane. Let $\mathbf{r} = (x, y)$ be the coordinates in a plane that is rotating at $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ relative to the inertial frame.

$$\begin{aligned}
\mathbf{v} &= \dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r} && (20) \\
\mathbf{a} &= \ddot{\mathbf{r}} + \boldsymbol{\Omega} \times \dot{\mathbf{r}} + \boldsymbol{\Omega} \times (\dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r}) && (21) \\
&= \ddot{\mathbf{r}} + 2\boldsymbol{\Omega} \times \dot{\mathbf{r}} - \Omega^2\, \mathbf{r} && (22) \\
\mathfrak{G} &= \frac{1}{2}\, m \left(\ddot{r}^2 + 4\ddot{\mathbf{r}} \cdot \boldsymbol{\Omega} \times \dot{\mathbf{r}} - 2\Omega^2\, \ddot{\mathbf{r}} \cdot \mathbf{r} \right) && (23) \\
m \left(\ddot{\mathbf{r}} + 2\boldsymbol{\Omega} \times \dot{\mathbf{r}} - \Omega^2 \mathbf{r} \right) &= \mathbf{F} && (24) \\
m\ddot{\mathbf{r}} &= \mathbf{F} - 2m\boldsymbol{\Omega} \times \dot{\mathbf{r}} + m\Omega^2 \mathbf{r} && (25)
\end{aligned}$$

which displays the Coriolis and centrifugal pseudo-forces.

The Rolling Penny

We have from the Appendix (particularly Example 2):

$$\mathfrak{G}_{\text{rot}} = \frac{1}{2}\, \left(I_1\, (\dot{\omega}_1^2 + \dot{\omega}_2^2) + I_3\, \dot{\omega}_3^2 \right) + (I_1\Omega_3 - I_3\omega_3)(\dot{\omega}_2\omega_1 - \dot{\omega}_1\omega_2) \tag{26}$$

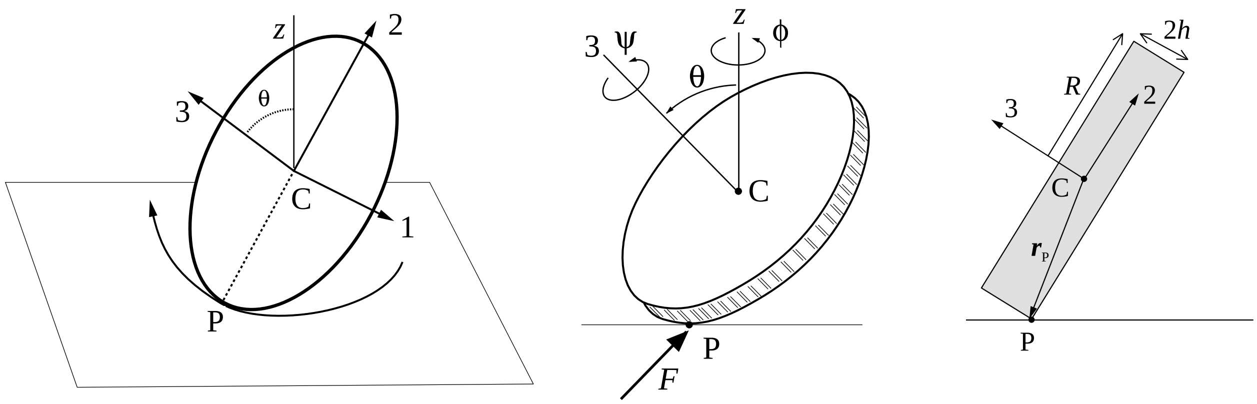

Figure 1: Coordinate frame for rolling disk. Axis 3 is in the direction of the disk's axle; 1 is always parallel to the plane and in the trailing direction if $\dot{\psi} > 0$; 2 points away from the contact point P.

We must add to this $\mathfrak{G}_{CM}$. Following `rolling.pdf` Eq. (7) and using $\mathbf{r}_P = -R\mathbf{e}_2$ we have:

$$\begin{aligned} \mathbf{A}_{CM} &= -\dot{\boldsymbol{\omega}} \times \mathbf{r}_P - \boldsymbol{\Omega} \times \boldsymbol{\omega} \times \mathbf{r}_P && (27) \\ &= R\left\{\dot{\boldsymbol{\omega}} \times \mathbf{e}_2 + \boldsymbol{\Omega} \times \boldsymbol{\omega} \times \mathbf{e}_2\right\} && (28) \\ &= R(-\dot{\omega}_3 + \Omega_2\omega_1,\ -\Omega_1\omega_1 - \Omega_3\omega_3,\ \dot{\omega}_1 + \Omega_2\omega_3) && (29) \end{aligned}$$

So:

$$\mathfrak{G}_{CM} = \frac{1}{2}\, MR^2 \left((\dot{\omega}_3 - \omega_2\omega_1)^2 + (\dot{\omega}_1 + \omega_2\omega_3)^2\right) \tag{30}$$

The resulting equations of motion (with $V = MgR\sin\theta$)

$$\begin{aligned} MR^2(\dot{\omega}_1 + \omega_2\omega_3) + I_1\,\dot{\omega}_1 - (I_1\Omega_3 - I_3\omega_3)\omega_2 &= -MgR\cos\theta && (31) \\ I_1\,\dot{\omega}_2 + (I_1\Omega_3 - I_3\omega_3)\omega_1 &= 0 && (32) \\ MR^2(\dot{\omega}_3 - \omega_2\omega_1) + I_3\,\dot{\omega}_3 &= 0 && (33) \end{aligned}$$

Making the usual rescaling: $I \leftarrow I/MR^2$, $g \leftarrow g/R$:

$$\begin{aligned} (I_1+1)\,\dot{\omega}_1 - I_1\Omega_3\omega_2 + (I_3+1)\omega_2\omega_3 &= -g\cos\theta && (34) \\ I_1\,\dot{\omega}_2 + (I_1\Omega_3 - I_3\omega_3)\omega_1 &= 0 && (35) \\ (I_3+1)\,\dot{\omega}_3 - \omega_2\omega_1 &= 0 && (36) \end{aligned}$$

$$\begin{pmatrix} (I_1+1)\,\dot{\omega}_1 \\ I_1\,\dot{\omega}_2 \\ (I_3+1)\,\dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} I_1\Omega_3\omega_2 - (I_3+1)\omega_2\omega_3 - g\cos\theta \\ (I_3\omega_3 - I_1\Omega_3)\omega_1 \\ \omega_2\omega_1 \end{pmatrix} \tag{37}$$

The Rolling Ring

Following Fig. 1 RHS, $\mathbf{r}_P = -R(\mathbf{e}_2 + h\mathbf{e}_3)$ (note that h has already been scaled: $h \leftarrow h/R$), we have:

$$\mathbf{A}_{CM} = -\dot{\boldsymbol{\omega}} \times \mathbf{r}_P - \boldsymbol{\Omega} \times \boldsymbol{\omega} \times \mathbf{r}_P \tag{38}$$

$$\mathbf{A}_{CM}/R = \dot{\boldsymbol{\omega}} \times (\mathbf{e}_2 + h\mathbf{e}_3) + \boldsymbol{\Omega} \times \boldsymbol{\omega} \times (\mathbf{e}_2 + h\mathbf{e}_3) \tag{39}$$

$$= \begin{pmatrix} -\dot{\omega}_3 + h\dot{\omega}_2 + \omega_1\omega_2 + h\omega_1\Omega_3 \\ -h\dot{\omega}_1 - \omega_1^2 - \omega_3\Omega_3 + h\omega_2\Omega_3 \\ \dot{\omega}_1 + \omega_2\omega_3 - h(\omega_1^2 + \omega_2^2) \end{pmatrix} \tag{40}$$

So:

$$\mathfrak{G}_{CM} = \frac{1}{2}\, MR^2 \big(\dot{\omega}_1^2(1+h^2) + h^2\dot{\omega}_2^2 + \dot{\omega}_3^2 + \\ 2\dot{\omega}_1(\omega_3 - h\omega_2)(\omega_2 + h\Omega_3) + 2\dot{\omega}_2\omega_1 h(\omega_2 + h\Omega_3) - 2\dot{\omega}_3(\dot{\omega}_2 h + \omega_1\omega_2 + h\omega_1\Omega_3)\big) \tag{41}$$

The resulting LHS equations of motion

$$\begin{pmatrix} MR^2\left(\dot{\omega}_1(1+h^2) + (h\omega_3 - \omega_2)(\omega 2 + h\Omega_3)\right) + I_1\,\dot{\omega}_1 - (I_1\Omega_3 - I_3\omega_3)\omega_2 \\ MR^2\left(h(-\dot{\omega}_3 + \dot{\omega}_2 h + \omega_1(\omega_2 + h\Omega_3))\right) + I_1\,\dot{\omega}_2 + (I_1\Omega_3 - I_3\omega_3)\omega_1 \\ MR^2\left(\dot{\omega}_3 - \dot{\omega}_2 h - \omega_1(\omega_2 + h\Omega_3)\right) + I_3\,\dot{\omega}_3 \end{pmatrix} \tag{42}$$

With $V = MgR(\sin\theta + h\cos\theta)$ the RHS is quite simple:

$$\begin{pmatrix} -MgR(\cos\theta - h\sin\theta) \\ 0 \\ 0 \end{pmatrix} \tag{43}$$

Scaling the variables as usual and following the division of LHS/RHS as in `rolling.pdf` Eq. (50):

$$\begin{pmatrix} (I_1 + 1 + h^2)\,\dot{\omega}_1 \\ (I_1 + h^2)\,\dot{\omega}_2 - h\dot{\omega}_3 \\ (I_3 + 1)\,\dot{\omega}_3 - h\dot{\omega}_2 \end{pmatrix} = \begin{pmatrix} (I_1 + 1 + h^2)\,\ddot{\theta} \\ (I_1 + h^2)\,(\ddot{\phi}\sin\theta + \dot{\phi}\cos\theta\dot{\theta}) - h(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\sin\theta\dot{\theta}) \\ (I_3 + 1)\,(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\sin\theta\dot{\theta}) - h(\ddot{\phi}\sin\theta + \dot{\phi}\cos\theta\dot{\theta}) \end{pmatrix} = \\ \begin{pmatrix} ((I_1 + h^2)\Omega_3 - (I_3 + 1)\omega_3 + h\omega_2)\omega_2 - h\omega_3\Omega_3 - g(\cos\theta - h\sin\theta) \\ (I_3\omega_3 - (I_1 + h^2)\Omega_3 - h\omega_2)\omega_1 \\ \omega_1(\omega_2 + h\Omega_3) \end{pmatrix} \tag{44}$$

Hurricane Balls

Following Fig. 2, $\mathbf{r}_P = -a(\sin\theta\,\mathbf{e}_2 + (1 + \cos\theta)\mathbf{e}_3)$

$$\mathbf{A}_{CM} = -\dot{\boldsymbol{\omega}} \times \mathbf{r}_P - \boldsymbol{\omega} \times \dot{\mathbf{r}}_P - \boldsymbol{\Omega} \times \boldsymbol{\omega} \times \mathbf{r}_P \tag{45}$$

$$\mathbf{A}_{CM}/a = \dot{\boldsymbol{\omega}} \times (\sin\theta\,\mathbf{e}_2 + (1 + \cos\theta)\mathbf{e}_3) + \boldsymbol{\omega} \times (\cos\theta\,\mathbf{e}_2 - \sin\theta\,\mathbf{e}_3)\omega_1 + \tag{46}$$

$$\boldsymbol{\Omega} \times \boldsymbol{\omega} \times (\sin\theta\,\mathbf{e}_2 + (1 + \cos\theta)\mathbf{e}_3) \tag{47}$$

$$= \begin{pmatrix} \dot{\omega}_2 + \omega_1\Omega_3 + (\dot{\omega}_2 + \omega_1(-\omega_3 + \Omega_3))\cos\theta - \dot{\omega}_3\sin\theta \\ -\dot{\omega}_1 + \omega_2\Omega_3 + (-\dot{\omega}_1 + \omega_2\Omega_3)\cos\theta - \omega_3\Omega_3\sin\theta \\ -\omega_1^2 - \omega_2^2 - \omega_2^2\cos\theta + (\dot{\omega}_1 + \omega_2\omega_3)\sin\theta \end{pmatrix} \tag{48}$$

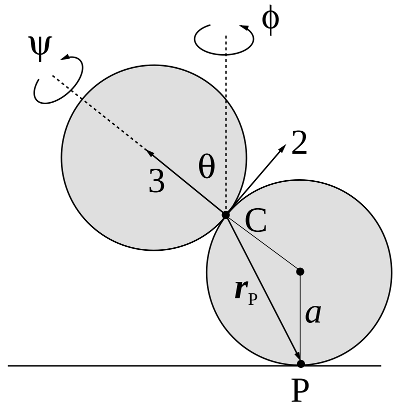

Figure 2: Coordinate system for Hurricane Balls

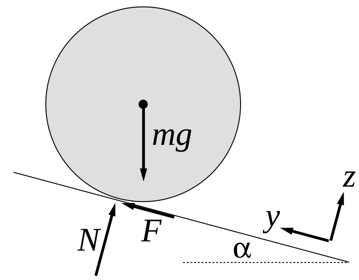

Figure 3: Coordinate system for rolling on a tilted plane

So:

$$\begin{aligned}\mathfrak{G}_{CM} = ma^2\Big(&2\dot{\omega}_1^2(1+\cos\theta) + \dot{\omega}_2^2(1+\cos\theta)^2 + \dot{\omega}_3^2\sin^2\theta + \\ &2\dot{\omega}_1(-\omega_2(\Omega_3(1+\cos\theta)^2 - \omega_3\sin^2\theta) - \sin\theta(\omega_1^2 + (\omega_2^2 - \omega_3\Omega_3)(1+\cos\theta))) + \\ &2\dot{\omega}_2(1+\cos\theta)(\omega_1\Omega_3(1+\cos\theta) - \omega_1\omega_3\cos\theta - \dot{\omega}_3\sin\theta) + \\ &2\dot{\omega}_3(\omega_1\sin\theta(\omega_3\cos\theta - \Omega_3(1+\cos\theta)))\Big)\end{aligned} \tag{49}$$

The resulting LHS equations of motion (scaling the variables as usual: $I \leftarrow I/2ma^2$ and letting $c = \cos\theta, s = \sin\theta$):

$$\begin{pmatrix} 2\dot{\omega}_1(1+c) - s(\omega_1^2 + (\omega_2^2 - \omega_3\Omega_3)(1+c)) - \omega_2(\Omega_3(1+c)^2 - \omega_3 s^2) + I_1\,\dot{\omega}_1 - (I_1\Omega_3 - I_3\omega_3)\omega_2 \\ \dot{\omega}_2(1+c)^2 + (1+c)(\omega_1\Omega_3(1+c) - \omega_1\omega_3 c - \dot{\omega}_3 s) + I_1\,\dot{\omega}_2 + (I_1\Omega_3 - I_3\omega_3)\omega_1 \\ -\dot{\omega}_2(1+c)s + \omega_1(\omega_3 c - \Omega_3(1+c))s + \dot{\omega}_3 s^2 + I_3\,\dot{\omega}_3 \end{pmatrix} \tag{50}$$

With $V = 2mga(1+\cos\theta)$, scaling by $2ma^2$ and $g \leftarrow g/a$ the RHS is quite simple:

$$\begin{pmatrix} g\sin\theta \\ 0 \\ 0 \end{pmatrix} \tag{51}$$

Following the division of LHS/RHS as in `rolling.pdf` Eq. (80–81), the RHS becomes:

$$\begin{pmatrix} s(g + \omega_1^2 + (\omega_2^2 - \omega_3\Omega_3)(1+c)) + \omega_2(\Omega_3(I_1 + (1+c)^2) - \omega_3(I_3 + s^2)) \\ -\omega_1(1+c)(\Omega_3(1+c) - \omega_3 c) - \omega_1(I_1\Omega_3 - I_3\omega_3) \\ -\omega_1(\omega_3 c - \Omega_3(1+c))s \end{pmatrix} \tag{52}$$

and the LHS:

$$\begin{pmatrix} \dot{\omega}_1(I_1 + 2(1+c)) \\ \dot{\omega}_2(I_1 + (1+c)^2) - \dot{\omega}_3(1+c)s \\ \dot{\omega}_3(s^2 + I_3) - \dot{\omega}_2(1+c)s \end{pmatrix} \tag{53}$$

Appendix: Finding $\mathfrak{G}_{\rm rot}$ for a Rigid Body

We seek a formula for $\mathfrak{G}_{\rm rot}$ calculated in a frame where the 123 axes are aligned with the principal axes. So

$$\int \left(r^2 \overleftrightarrow{\mathbf{1}} - \vec{\mathbf{r}}\,\vec{\mathbf{r}}\right) dm = \begin{pmatrix} I_1 & 1 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{54}$$

hence, for example, $\int xy\, dm = 0$ and $\int (y^2+z^2)\, dm = I_1$.

This principal axes frame is rotating at $\mathbf{\Omega}$ relative to the inertial frame, the rigid body has angular velocity $\boldsymbol{\omega}$, and we let $\mathbf{r}, \mathbf{u}, \mathbf{a}$ denote respectively the location, velocity, and acceleration of a piece of the rigid body relative to the CM.

$$\begin{aligned}
\mathbf{u} &= \boldsymbol{\omega}\times\mathbf{r} = \dot{\mathbf{r}} + \mathbf{\Omega}\times\mathbf{r} \qquad \text{hence: } \dot{\mathbf{r}} = (\boldsymbol{\omega}-\mathbf{\Omega})\times\mathbf{r} && (55)\\
\mathbf{a} &= \boldsymbol{\omega}\times\dot{\mathbf{r}} + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \mathbf{\Omega}\times\boldsymbol{\omega}\times\mathbf{r} && (56)\\
&= \boldsymbol{\omega}\times(\boldsymbol{\omega}-\mathbf{\Omega})\times\mathbf{r} + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \mathbf{\Omega}\times\boldsymbol{\omega}\times\mathbf{r} && (57)\\
&= \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r} + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \mathbf{\Omega}\times\boldsymbol{\omega}\times\mathbf{r} - \boldsymbol{\omega}\times\mathbf{\Omega}\times\mathbf{r} && (58)\\
&= (\boldsymbol{\omega}\cdot\mathbf{r})\,\boldsymbol{\omega} - \omega^2\,\mathbf{r} + \dot{\boldsymbol{\omega}}\times\mathbf{r} + (\mathbf{\Omega}\cdot\mathbf{r})\,\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\mathbf{r})\,\mathbf{\Omega} && (59)\\
&= (\boldsymbol{\omega}\cdot\mathbf{r})\,\boldsymbol{\omega} - \omega^2\,\mathbf{r} + \dot{\boldsymbol{\omega}}\times\mathbf{r} - \mathbf{r}\times\mathbf{\Omega}\times\boldsymbol{\omega} && (60)\\
&= (\boldsymbol{\omega}\cdot\mathbf{r})\,\boldsymbol{\omega} - \omega^2\,\mathbf{r} - \mathbf{r}\times(\dot{\boldsymbol{\omega}} + \mathbf{\Omega}\times\boldsymbol{\omega}) && (61)\\
&\equiv (\boldsymbol{\omega}\cdot\mathbf{r})\,\boldsymbol{\omega} - \omega^2\,\mathbf{r} - \mathbf{r}\times\boldsymbol{\phi} && (62)
\end{aligned}$$

Two important points: (A) only $\boldsymbol{\phi}$ contains acceleration $\dot{\boldsymbol{\omega}}$, so in calculating $\mathfrak{G}_{\rm rot}$ we can drop terms that do not contain $\boldsymbol{\phi}$ and (B) when dotted with itself the term $\mathbf{r}\times\boldsymbol{\phi}$ is nicely connected with $\overleftrightarrow{\mathbf{I}}$:

$$(\mathbf{r}\times\boldsymbol{\phi})\cdot(\mathbf{r}\times\boldsymbol{\phi}) = \boldsymbol{\phi}\cdot(\mathbf{r}\times\boldsymbol{\phi}\times\mathbf{r}) = \boldsymbol{\phi}\cdot(r^2\,\overleftrightarrow{\mathbf{1}} - \vec{\mathbf{r}}\,\vec{\mathbf{r}})\cdot\boldsymbol{\phi} \tag{63}$$

So

$$\int (\mathbf{r}\times\boldsymbol{\phi})^2\, dm = \boldsymbol{\phi}\cdot\overleftrightarrow{\mathbf{I}}\cdot\boldsymbol{\phi} \tag{64}$$

The non-zero cross term in $\mathbf{a}\cdot\mathbf{a}$ that includes $\boldsymbol{\phi}$: $-2(\boldsymbol{\omega}\cdot\mathbf{r})\,\boldsymbol{\omega}\cdot(\mathbf{r}\times\boldsymbol{\phi})$ looks to be a mess, but note that the $\mathbf{r}$ component in $\boldsymbol{\omega}\cdot(\mathbf{r}\times\boldsymbol{\phi})$ must match the $\mathbf{r}$ component in $\boldsymbol{\omega}\cdot\mathbf{r}$ as non-matching terms will vanish when integrated as, e.g., $\int xy\, dm = 0$. Dropping terms that will vanish on integration yields:

$$-2(\boldsymbol{\omega}\cdot\mathbf{r})\begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ x_1 & x_2 & x_3 \\ \phi_1 & \phi_2 & \phi_3 \end{vmatrix} = \begin{array}{l} -\,2(\omega_2\omega_1\phi_3 x_2^2 + \omega_3\omega_2\phi_1 x_3^2 + \omega_1\omega_3\phi_2 x_1^2) \\ +\,2(\omega_3\omega_1\phi_2 x_3^2 + \omega_1\omega_2\phi_3 x_1^2 + \omega_2\omega_3\phi_1 x_2^2) \end{array} \tag{65}$$

$$= -2\omega_2\omega_1\phi_3(x_2^2 - x_1^2) - 2\omega_3\omega_2\phi_1(x_3^2 - x_2^2) - 2\omega_1\omega_3\phi_2(x_1^2 - x_3^2) \tag{66}$$

$$\rightarrow -2\omega_2\omega_1\dot{\omega}_3(I_1 - I_2) - 2\omega_3\omega_2\dot{\omega}_1(I_2 - I_3) - 2\omega_1\omega_3\dot{\omega}_2(I_3 - I_1) \tag{67}$$

So

$$\mathfrak{G}_{\rm rot} = \frac{1}{2}\,\boldsymbol{\phi}\cdot\overleftrightarrow{\mathbf{I}}\cdot\boldsymbol{\phi} + \omega_2\omega_1\dot{\omega}_3(I_2 - I_1) + \omega_3\omega_2\dot{\omega}_1(I_3 - I_2) + \omega_1\omega_3\dot{\omega}_2(I_1 - I_3) \tag{68}$$

Example 1: In the general case $I_1 \neq I_2 \neq I_3$ we must evaluate in the body fixed frame so $\boldsymbol{\Omega} = \boldsymbol{\omega}$ and $\boldsymbol{\phi} = \dot{\boldsymbol{\omega}}$, so

$$\begin{aligned}\mathfrak{G}_{\mathrm{rot}} &= \frac{1}{2}\left(I_1\,\dot{\omega}_1^2 + I_2\,\dot{\omega}_2^2 + I_3\,\dot{\omega}_3^2\right) + \\ &\quad \omega_2\omega_1\dot{\omega}_3(I_2 - I_1) + \omega_3\omega_2\dot{\omega}_1(I_3 - I_2) + \omega_1\omega_3\dot{\omega}_2(I_1 - I_3) \end{aligned} \tag{69}$$

The equations of motion are Euler's equations:

$$I_1\,\dot{\omega}_1 + \omega_3\omega_2(I_3 - I_2) = \Gamma_1 \tag{70}$$
$$I_2\,\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = \Gamma_2 \tag{71}$$
$$I_3\,\dot{\omega}_3 + \omega_2\omega_1(I_2 - I_1) = \Gamma_3 \tag{72}$$

where $\boldsymbol{\Gamma}$ is the torque in the body-fixed frame.

Example 2: Consider a top ($I_1 = I_2 \neq I_3$), where the calculational frame has the top spinning along the 3-axis (inclined at θ from z axis) with the 1-axis horizontal. We then have:

$$\boldsymbol{\Omega} = \left(\dot{\theta},\ \dot{\phi}\sin\theta,\ \dot{\phi}\cos\theta\right) \tag{73}$$
$$\boldsymbol{\omega} = \left(\dot{\theta},\ \dot{\phi}\sin(\theta),\ \dot{\phi}\cos(\theta) + \dot{\psi}\right) \tag{74}$$

(cf. `rolling.pdf` Eqs. (13-16)) Note: $\boldsymbol{\omega} = \boldsymbol{\Omega} + (0,\ 0,\ \omega_3 - \Omega_3)$

$$\boldsymbol{\phi} = \dot{\boldsymbol{\omega}} + \boldsymbol{\Omega} \times \boldsymbol{\omega} = \dot{\boldsymbol{\omega}} + \boldsymbol{\Omega} \times (0,\ 0,\ \omega_3 - \Omega_3) = \dot{\boldsymbol{\omega}} + (\omega_2,\ -\omega_1,\ 0)\{\omega_3 - \Omega_3\} \tag{75}$$

So

$$\begin{aligned}\mathfrak{G}_{\mathrm{rot}} &= \frac{1}{2}\left(I_1\,(\dot{\omega}_1^2 + \dot{\omega}_2^2) + I_3\,\dot{\omega}_3^2\right) + I_1\dot{\omega}_1\omega_2\{\omega_3 - \Omega_3\} - I_1\dot{\omega}_2\omega_1\{\omega_3 - \Omega_3\} + \\ &\quad \omega_3\omega_2\dot{\omega}_1(I_3 - I_1) + \omega_1\omega_3\dot{\omega}_2(I_1 - I_3) \end{aligned} \tag{76}$$
$$= \frac{1}{2}\left(I_1\,(\dot{\omega}_1^2 + \dot{\omega}_2^2) + I_3\,\dot{\omega}_3^2\right) + I_1\Omega_3(\dot{\omega}_2\omega_1 - \dot{\omega}_1\omega_2) + I_3\omega_3(\omega_2\dot{\omega}_1 - \omega_1\dot{\omega}_2) \tag{77}$$
$$= \frac{1}{2}\left(I_1\,(\dot{\omega}_1^2 + \dot{\omega}_2^2) + I_3\,\dot{\omega}_3^2\right) + (I_1\Omega_3 - I_3\omega_3)(\dot{\omega}_2\omega_1 - \dot{\omega}_1\omega_2) \tag{78}$$

The resulting equations of motion:

$$I_1\,\dot{\omega}_1 - (I_1\Omega_3 - I_3\omega_3)\omega_2 = Mg\ell\sin\theta \tag{79}$$
$$I_1\,\dot{\omega}_2 + (I_1\Omega_3 - I_3\omega_3)\omega_1 = 0 \tag{80}$$
$$I_3\,\dot{\omega}_3 = 0 \tag{81}$$

From the last equation we conclude ω_3 =constant, which relates to p_ψ =constant in the usual treatment. Substituting our Euler angle expression for $\boldsymbol{\omega}$ into the second equation:

$$I_1\,(\ddot{\phi}\sin\theta + 2\dot{\phi}\cos\theta\dot{\theta}) - I_3\omega_3\dot{\theta} = 0 \tag{82}$$
$$\frac{d}{dt}\left\{I_1\dot{\phi}\sin^2\theta + I_3\omega_3\cos\theta\right\} = 0 \tag{83}$$
$$I_1\dot{\phi}\sin^2\theta + I_3\omega_3\cos\theta \equiv p_\phi = \text{constant} \tag{84}$$

where in the second line we used $\sin\theta$ as an integrating factor.

If the first equation is multiplied by ω_1 and the second by ω_2 and the two are added:

$$I_1\,(\omega_1\dot\omega_1 + \omega_2\dot\omega_2) \;=\; Mg\ell\sin\theta\dot\theta \tag{85}$$

$$\frac{d}{dt}\left\{\frac{1}{2}\,I_1\,(\omega_1^2+\omega_2^2) + Mg\ell\cos\theta\right\} \;=\; 0 \tag{86}$$

$$\frac{1}{2}\,I_1\,(\omega_1^2+\omega_2^2) + Mg\ell\cos\theta \;\equiv\; E_\perp = \text{constant} \tag{87}$$

which is conservation of energy (aside from the conserved energy associated with ω_3). The above are the usual starting point to describe gyroscopic motion.

Example 3: In the case of a sphere $I_1 = I_2 = I_3 \equiv I$; any coordinate system will have principal axes aligned with coordinate axes so use the initial inertial frame ($\mathbf{\Omega} = \mathbf{0}$), then:

$$\mathfrak{G}_{\text{rot}} = \frac{1}{2}\,I\left(\dot\omega_1^2+\dot\omega_2^2+\dot\omega_3^2\right) \tag{88}$$

Consider the rolling without slipping motion of a sphere on a plane tilted at an angle α with the y coordinate pointing uphill, z perpendicular to the plane (see Fig. 3). So $V = Mg\sin\alpha\; y$. We have:

$$\mathfrak{G} = \frac{1}{2}\,M\left(\ddot x^2+\ddot y^2\right) + \frac{1}{2}\,I\left(\dot\omega_1^2+\dot\omega_2^2+\dot\omega_3^2\right) \tag{89}$$

The rolling without slipping condition gives:

$$\mathbf{v}_{CM} \;=\; -R\hat{\mathbf{z}}\times\boldsymbol{\omega} \tag{90}$$

$$\mathbf{a}_{CM} \;=\; -R\hat{\mathbf{z}}\times\dot{\boldsymbol{\omega}} \tag{91}$$

$$\ddot x_{CM} \;=\; R\dot\omega_2 \tag{92}$$

$$\ddot y_{CM} \;=\; -R\dot\omega_1 \tag{93}$$

$$\mathfrak{G} \;=\; \frac{1}{2}\,(M + I/R^2)\,\left(\ddot x^2+\ddot y^2\right) + \frac{1}{2}\,I\,\dot\omega_3^2 \tag{94}$$

The resulting equations of motion:

$$(M + I/R^2)\,\ddot x \;=\; 0 \tag{95}$$

$$(M + I/R^2)\,\ddot y \;=\; -Mg\sin\alpha \tag{96}$$

$$I\,\dot\omega_3 \;=\; 0 \tag{97}$$

We can compare this to the Newtonian solution:

$$M\mathbf{a}_{CM} \;=\; -Mg\sin\alpha\;\hat{\mathbf{y}} + \mathbf{F} \tag{98}$$

$$\mathbf{a}_{CM} \;=\; -R\hat{\mathbf{z}}\times\dot{\boldsymbol{\omega}} \tag{99}$$

$$I\dot{\boldsymbol{\omega}} \;=\; -R\hat{\mathbf{z}}\times\mathbf{F} \tag{100}$$

$$\;=\; -R\hat{\mathbf{z}}\times(M\mathbf{a}_{CM} + MRg\sin\alpha\;\hat{\mathbf{y}}) \tag{101}$$

$$\;=\; MR^2(\hat{\mathbf{z}}\times\hat{\mathbf{z}}\times\dot{\boldsymbol{\omega}}) + MRg\sin\alpha\;\hat{\mathbf{x}} \tag{102}$$

$$\;=\; MR^2(\dot\omega_3\hat{\mathbf{z}} - \dot{\boldsymbol{\omega}}) + MRg\sin\alpha\;\hat{\mathbf{x}} \tag{103}$$

$$(I + MR^2)\,\dot\omega_1 \;=\; MgR\sin\alpha \tag{104}$$

$$(I + MR^2)\,\dot\omega_2 \;=\; 0 \tag{105}$$

$$I\,\dot\omega_3 \;=\; 0 \tag{106}$$