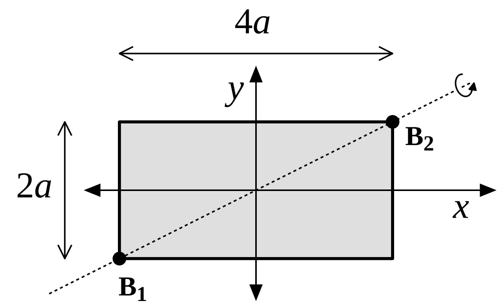
Complete 4 of these 5 problems

1. Consider a flat, uniform rectangular plate of mass: M, and with sides: $4a \times 2a$ that at this instant lies in the z plane.

   (a) The moment of inertia tensor of this plate (about its CM) is given by:

   $$I_P = \frac{1}{3}Ma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

   Calculate the integral that is I_{yy} to confirm the above entry. Explain why the integral that is I_{xy} is zero.



   (b) The plate is rotating at angular velocity $\boldsymbol{\omega}$ about the (fixed) axis shown (a diagonal). Calculate the angular momentum (about its CM) at this instant. Draw the resulting vector directly on the diagram.

   (c) Report the direction and magnitude of the torque (about the CM) on the plate.

   (d) $\mathbf{B_1}$ and $\mathbf{B_2}$ are the bearings for the rotation axis. Directly on the diagram above, show the direction of any forces applied to the plate at those locations.

2. In the body-fixed frame the axes (123) are aligned with the principal axes and the body's symmetry axis is coincident with the 3-axis. We seek the direction of the body-fixed 3-axis in the inertial ('space') frame. At this instant, the orientation of the body-fixed frame is described by three Euler angles: ϕ, θ, ψ. The connection between the inertial frame and the body-fixed frame is made through three successive rotations: (1) a rotation about the z axis by the angle ϕ connecting the inertial frame to frame$''$ [(inertial) = $\mathcal{M}_\phi\cdot$(frame$''$)]

   $$\mathcal{M}_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   (2) a rotation about the y'' axis by θ connecting frame$''$ to frame$'$ [(frame$''$) = $\mathcal{M}_\theta\cdot$(frame$'$)]

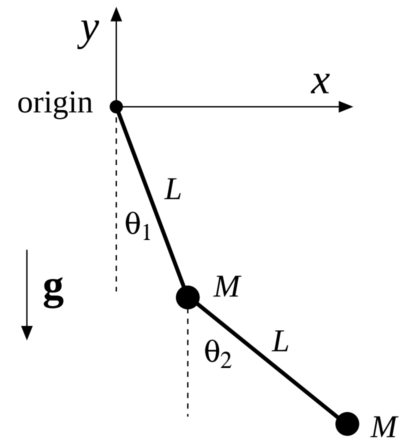
   $$\mathcal{M}_\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

   (3) a rotation about the z' axis by ψ connecting the frame$'$ to the body-fixed frame (123):

   $$\mathcal{M}_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   Find an expression for the direction of the body-fixed 3-axis in the inertial frame.

3. On the last exam (and even more extensively in chapter 11), we considered the equal-mass (M), equal-length (L), double pendulum. The pendulum consists of two identical pendulums with the second pendulum attached to the first. The (x, y) locations of the two masses ($\mathbf{r}_1$ & $\mathbf{r}_2$), the kinetic energy (KE), and the gravitational potential energy (PE) are given by:



$$\begin{aligned}
\mathbf{r}_1 &= (L\sin\theta_1, -L\cos\theta_1) \\
\mathbf{r}_2 &= (L(\sin\theta_1 + \sin\theta_2), -L(\cos\theta_1 + \cos\theta_2)) \\
\text{KE} &= \frac{1}{2}\,ML^2\left(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)\right) \\
\text{PE} &= -MLg\,(2\cos\theta_1 + \cos\theta_2)
\end{aligned}$$

Clearly the stable equilibrium from which the normal modes will be small perturbations is $\theta_1 = \theta_2 = 0$.

(a) $\mathcal{K}$ is the matrix of second partial derivatives of PE evaluated at the equilibrium position. Via Taylors expansion it allows you to approximate the PE for small deviations from the stable equilibrium. Calculate it!

(b) The symmetric matrix $\mathcal{M}$ similarly allows approximate calculation of the KE for small deviations about the stable equilibrium. Calculate it!

(c) With all of the above the Lagrangian can be approximated by:

$$L = \frac{1}{2}\,\dot{\mathbf{\Theta}}^T \cdot \mathcal{M} \cdot \dot{\mathbf{\Theta}} - \frac{1}{2}\,\mathbf{\Theta}^T \cdot \mathcal{K} \cdot \mathbf{\Theta} = \frac{1}{2}\sum_{i,j}\left(\dot{\theta}_i \mathcal{M}_{ij}\dot{\theta}_j - \theta_i \mathcal{K}_{ij}\theta_j\right)$$

where the vector $\mathbf{\Theta} = (\theta_1, \theta_2)$. Show that the resulting equations of motions are:

$$\mathcal{M} \cdot \ddot{\mathbf{\Theta}} = -\mathcal{K} \cdot \mathbf{\Theta}$$

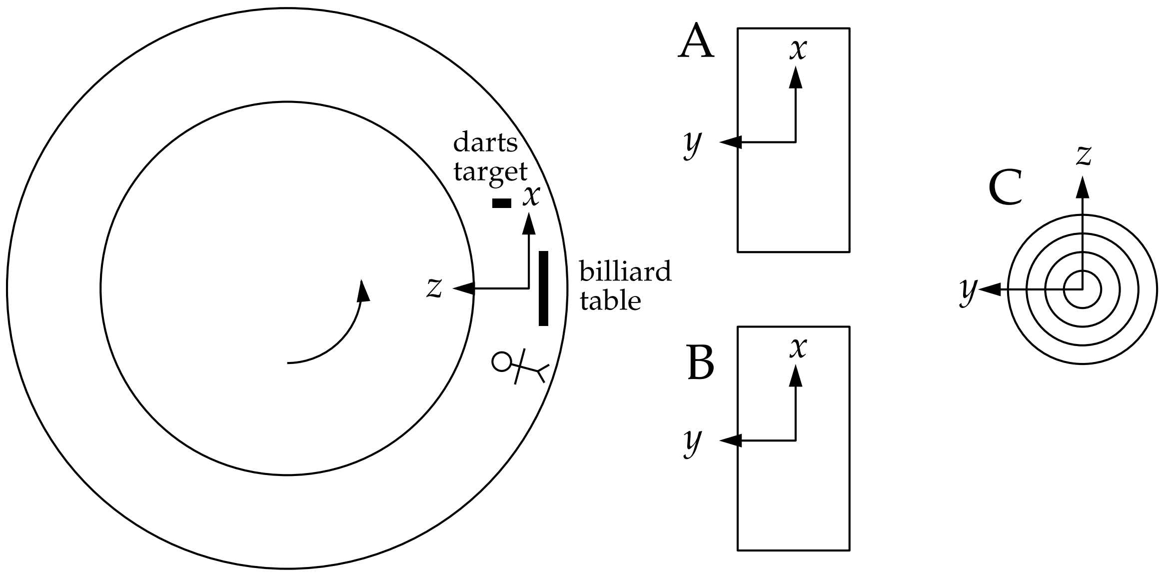
That is the equation of motion for the k^{th} component of $\mathbf{\Theta}$ (i.e., θ_k), is:

$$\sum_{j=1}^{n} \mathcal{M}_{kj}\ddot{\theta}_j = -\sum_{j=1}^{n} \mathcal{K}_{kj}\theta_j$$

(d) If (contrary to fact) $\mathcal{M}$ and $\mathcal{K}$ were:

$$\mathcal{M} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \qquad \text{and} \qquad \mathcal{K} = \begin{bmatrix} 3 & -3 \\ -3 & 28 \end{bmatrix}$$

show that $\mathbf{\Theta} = (6, 1)e^{i\omega t}$ would be a normal mode with $\omega^2 = 5/2$.



4. Imagine your experience in a 'big wheel' space station whose living apace is a torus with radius $R = 10$ m and is rotated so the centrifugal force is equivalent to the force of gravity on the surface of the Earth: mg. Your x axis aligned with the wheel's circumference and points in the direction of the wheel's tangential velocity, your z axis points toward the center of the wheel (i.e., in the apparent 'up' direction) and your y axis points to your left.

   (a) Calculate/report the magnitude and direction of the wheel's angular frequency vector $\boldsymbol{\omega}$.

   (b) You are playing billiards on a table which, when viewed from above, is as shown above. Directly on the diagram **A** show a billiard ball's path if it is hit (given an initial velocity) in exactly the positive x direction. Name all the non-zero forces in this rotating frame that determine the ball's path and in words or pictures show their directions.

   (c) Directly on the diagram **B** show a billiard ball's path if it is hit (given an initial velocity) in exactly the positive y direction. Name all the non-zero forces in this rotating frame that determine the ball's path and in words or pictures show their directions.

   (d) As a skilled dart thrower on Earth you hit the bull's eye every time. The space station's target lies directly ahead of you in the positive x direction. Throwing exactly as on Earth, mark on the given target **C** where your dart hits on the space station's target. Name all the forces in this rotating frame that affect the dart's flight and in words or pictures show their directions.

   (e) As viewed by an observer in a non-rotating, inertial frame, describe the dart's path. Name all the forces that cause the motion and in words or pictures show their directions.

5. Consider the 'normal modes' of a satellite at the L4 Lagrange point (this is the stable three body solution that forms an equilateral triangle with the other two objects, e.g., Earth and Moon). We are working in a rotating coordinate system (so a Coriolis force is present and a centrifugal potential has been incorporated into the potential energy) and are using dimensionless quantities. All three objects are in a plane so $\mathbf{r} = (x, y)$ is just two dimensional. As usual we can determine the $\mathcal{K}$ matrix by calculating the partial derivatives of the potential energy $U(x, y)$ evaluated at the equilibrium point:

$$\frac{\partial^2 U}{\partial r_i \partial r_j} = \mathcal{K} = \begin{pmatrix} -\alpha^2 & -\frac{3}{2}\alpha(1-2e) \\ -\frac{3}{2}\alpha(1-2e) & -3\alpha^2 \end{pmatrix}$$

where e is the mass ratio of the two massive objects and $\alpha^2 = \frac{3}{4}$. The equations of motion about the L4 equilibrium point are:

$$\frac{1}{(2\pi)^2}\,\ddot{\mathbf{r}} = -\mathcal{K}\cdot\mathbf{r} + \frac{1}{\pi}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\cdot\dot{\mathbf{r}}$$

Notice the dot on the rightmost $\mathbf{r}$. Report why this term is present. As usual we seek an oscillating solution of the form $\mathbf{r} = \mathbf{a}e^{i\omega t}$ where $\mathbf{a}$ is a constant vector and ω is the unknown angular frequency. Plug this candidate solution into the above equation; you may find it convenient to use the variable: $z = \omega/(2\pi)$ instead of ω. Rearrange the result to show $\mathbf{a}$ must be a solution to a set of homogeneous (i.e., 0=rhs) linear equations. Write down the resulting polynomial z must satisfy if a nontrivial solution to the homogeneous linear equations exists. FYI: with simplification (which is not required for your answer) that polynomial is:

$$z^4 - z^2 + \frac{27}{4}\,e(1-e) = 0$$

which produces $\omega \in \mathbb{R}$ only if $27e(1-e) < 1$.