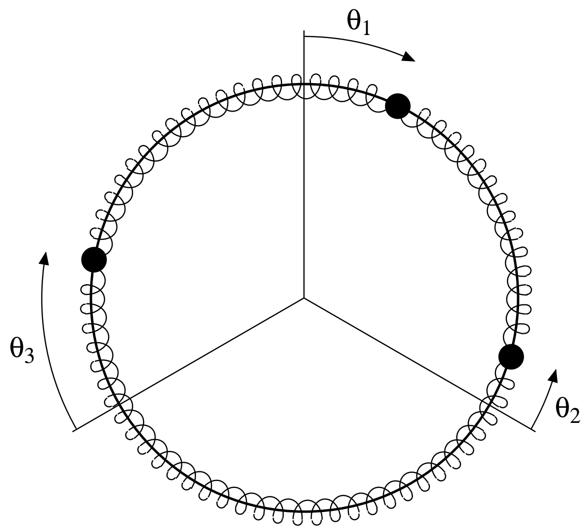
4. A simple model of a ring molecule consists of three equal masses m which slide without friction on a fixed circular wire with radius R. The masses are connected by identical springs with spring constant k. The angular positions of the masses $(\theta_1, \theta_2, \theta_3)$ are measured from a rest position in the clockwise sense as shown below (note that in the below diagram θ_2 would be negative).



(a) Find the Lagrangian.

(b) Show that the mass matrix $\mathcal{M}$ and the spring constant matrix $\mathcal{K}$ are as shown below:

$$\mathcal{M} = mR^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathcal{K} = kR^2 \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

(c) The Euler equations for this system are:

$$mR^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} = -kR^2 \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

or, defining the vector $\boldsymbol{\Theta} = (\theta_1, \theta_2, \theta_3)$

$$\mathcal{M} \cdot \ddot{\boldsymbol{\Theta}} = -\mathcal{K} \cdot \boldsymbol{\Theta}$$

if we seek a periodic solution $\boldsymbol{\Theta} = \mathbf{v}\, e^{i\omega t}$, (where $\mathbf{v}$ is a constant vector and ω is the constant angular oscillation frequency) we have:

$$\omega^2 \mathcal{M} \cdot \mathbf{v} = \mathcal{K} \cdot \mathbf{v}$$

Show that the vector $\mathbf{v} = (1, 1, 1)$ satisfies this equation for $\omega = 0$.

(d) Show that the vector $\mathbf{v} = (0, 1, -1)$ satisfies this equation and find the corresponding ω.