As described in the textbook, Euler Angles are a way to specify the configuration of a 3d object. Starting from a fixed configuration the desired configuration is obtained by a three step process:

1. rotation about the z axis by an angle ϕ
2. rotation about the x' axis[1] (i.e., the rotated x axis) by an angle θ
3. rotation about the z'' axis (i.e., the doubly rotated z axis which, in the end, is the body axis 3) by an angle ψ

I strongly recommend looking at the Wiki visualizations (`Euler.gif`, author Juansempere; also copied to the class web site) to appreciate these rotations. I hope it is clear that almost certainly the object did not achieve its configuration by exactly these three rotations just as it's unlikely that an object reached a particular position by successive motions in the x, y and z directions. We are recording configuration not history.

The body-fixed frame (123) with principal axes aligned with the frame is most convenient for calculation; but we often need to know what a body-fixed vector looks like in the inertial frame (xyz). We define matrices to reverse the above three steps:

$$\mathcal{M}_\phi = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

$$\mathcal{M}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{2}$$

$$\mathcal{M}_\psi = \begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

where:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathcal{M}_\phi \mathcal{M}_\theta \mathcal{M}_\psi \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{4}$$

Note: To make the reverse transformation (i.e., $(x, y, z) \to (x_1, x_2, x_3)$) you would apply the inverse matrices in the reverse order to (x, y, z). The inverse matrices are easily generated by negating the angle (e.g., $\theta \to -\theta$) or taking the matrix transpose.

We begin by finding the relation between $\dot\phi, \dot\theta, \dot\psi$ and $\boldsymbol{\omega}$ (in the body-fixed frame).

[1]This is the convention of Goldstein's *Classical Mechanics* and Wiki; our textbook makes this second rotation about the y' axis with the warning that it is not standard. I'm going here with the standard

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix} + \mathcal{M}_\psi^{-1} \begin{pmatrix} \dot\theta \\ 0 \\ 0 \end{pmatrix} + \mathcal{M}_\psi^{-1}\mathcal{M}_\theta^{-1} \begin{pmatrix} 0 \\ 0 \\ \dot\phi \end{pmatrix} \tag{5}$$

$$= \left(\dot\phi \sin(\psi)\sin(\theta) + \dot\theta\cos(\psi),\ \dot\phi\cos(\psi)\sin(\theta) - \dot\theta\sin(\psi),\ \dot\phi\cos(\theta) + \dot\psi\right) \tag{6}$$

Given $\boldsymbol{\omega}$ in the body-fixed frame it's easy (for *Mathematica*) to calculate the kinetic energy:

$$T = \frac{1}{2}\,\boldsymbol{\omega}\cdot \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \cdot \boldsymbol{\omega} \tag{7}$$

$$= \frac{1}{2}\, I_1 \left(\dot\phi^2 \sin^2(\theta) + \dot\theta^2\right) + \frac{1}{2}\, I_3 \left(\dot\phi\cos(\theta) + \dot\psi\right)^2 \tag{8}$$

The problem at hand is *free* precession…no external forces or potential energy; the Lagrangian is just the kinetic energy T. Notice that ϕ and ψ are cyclic (a.k.a., ignorable) coordinates so the corresponding canonical (a.k.a., generalized) momenta are constants:

$$p_\psi = \frac{\partial T}{\partial \dot\psi} = I_3\left(\dot\phi\cos(\theta) + \dot\psi\right) \tag{9}$$

$$p_\phi = \frac{\partial T}{\partial \dot\phi} = I_3\cos(\theta)\left(\dot\phi\cos(\theta) + \dot\psi\right) + I_1\,\dot\phi\sin^2(\theta) = p_\psi\cos(\theta) + I_1\,\dot\phi\sin^2(\theta) \tag{10}$$

Comparing to Eq. (6), see that $p_\psi = L_3$ (i.e., the angular momentum in the body-fixed z direction); at the end of this document we discovery $p_\phi = L_z$ (i.e., the angular momentum in the inertial frame z direction). Using these (constant) momenta we can rewrite the kinetic energy much as in a Hamiltonian (but we will leave $\dot\theta$ alone):

$$T = \frac{1}{2}\, I_1\,\dot\theta^2 + \frac{(p_\phi - p_\psi\cos(\theta))^2}{2I_1\sin^2(\theta)} + \frac{p_\psi^2}{2I_3} = \frac{1}{2}\, I_1\,\dot\theta^2 + V(\theta)$$

This expression now just involves constants and θ and $\dot\theta$; furthermore it is itself a constant. The usual logic of 1d conservation of energy applies to θ: turning points, equilibrium points, etc. In particular the minimum of $V(\theta)$ must be an equilibrium point where $\dot\theta = 0$. Working in terms of $c = \cos\theta$ note:

$$V(c) \propto \frac{(p_\phi - p_\psi c)^2}{1 - c^2} + \text{constant}$$

and $V' = 0$ has two solutions: $c = p_\phi/p_\psi$ and $c = p_\psi/p_\phi$ The first solution results in $\dot\phi = 0$ in addition to $\dot\theta = 0$. Applying those results to $\boldsymbol{\omega}$ see that $\boldsymbol{\omega}$ (and hence $\mathbf{L}$) are entirely along the body-fixed 3 axis. This is an object spinning in space with no additional motion. The kinetic energy is simply: $p_\psi^2/(2I_3)$—the kinetic energy of rotation just about the body-fixed 3 axis.

The second solution is more interesting. Using the constant values of $p_\psi, p_\phi, \cos\theta$ find the values of $\dot\phi$ and $\dot\psi$:

```
Solve[{Pphi==Ppsi Ppsi/Pphi + dphi I1 (1- (Ppsi/Pphi)^2),
Ppsi== I3 (dpsi + dphi (Ppsi/Pphi))},{dpsi,dphi}]

                     (I1 - I3) Ppsi          Pphi
Out[20]= {{dpsi -> --------------, dphi -> ----}}
                        I1 I3                I1
```

Thus a free body moves with

$$\theta = \theta_0 \tag{11}$$

$$\phi = \dot{\phi}_0 \, t = \frac{I_3 \omega_3}{I_1 \cos\theta_0} \, t \tag{12}$$

$$\psi = -\cos\theta_0 \, \frac{I_3 - I_1}{I_3} \, \dot{\phi}_0 \, t = -\,\frac{I_3 - I_1}{I_1} \, \omega_3 \, t \tag{13}$$

solves the equations of motion. Note that (θ, ϕ) define the direction of the body-fixed 3 axis; evidently it is inclined (at θ_0) and rotating at rate $\dot{\phi}_0$. In the body frame,

$$\boldsymbol{\omega} = \left(\frac{p_\phi \sin\theta_0}{I_1} \, \sin\psi, \; \frac{p_\phi \sin\theta_0}{I_1} \, \cos\psi, \; \frac{p_\psi}{I_3} \right) \tag{14}$$

i.e., ω_3 has a constant value of p_ψ / I_3 while $\boldsymbol{\omega}_\perp$ is rotating at rate $\dot{\psi}$ and has constant magnitude $p_\phi \sin\theta_0 / I_1$.

If we transform **L** from the body-fixed frame back into the inertial frame and substitute in the now know values for $\dot{\psi}, \dot{\phi}$ and $\dot{\theta} = 0$.

```
mphi.mtheta.mpsi.L
Simplify[%]
% /. {dphi->Pphi/I1, dpsi->Cos[theta](I1/I3-1)Pphi/I1,dtheta->0}
Simplify[%]

Out[24]= {0, 0, Pphi}
```

We conclude that this solution has **L** in the inertial frame aligned with the z axis.

As stated above the fact that $L_z = p_\phi$ is true in general:

```
mphi.mtheta.mpsi.L
Collect[%[[3]],{I1,I3},Simplify]
                                                                  2
Out[28]= I3 Cos[theta] (dpsi + dphi Cos[theta]) + dphi I1 Sin[theta]
```

where you'll notice this result is exactly p_ϕ