$$T = \frac{1}{2}\,\boldsymbol{\omega}\cdot\begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix}\cdot\boldsymbol{\omega} \tag{1}$$

$$= \frac{1}{2}\,I_1\left(\dot\phi^2\sin^2(\theta)+\dot\theta^2\right)+\frac{1}{2}\,I_3\left(\dot\phi\cos(\theta)+\dot\psi\right)^2 \tag{2}$$

The problem at hand is a spinning top with gravitational potential energy $mgR\cos\theta$. (R is the distance between the pivot-point—the origin—and the center of mass.) While Lagrangian is now a bit more than just the kinetic energy T, ϕ and ψ are not in the PE and hence remain cyclic (a.k.a., ignorable) coordinates so the corresponding canonical (a.k.a., generalized) momenta are constants:

$$p_\psi = \frac{\partial T}{\partial\dot\psi} = I_3\left(\dot\phi\cos(\theta)+\dot\psi\right) \tag{3}$$

$$p_\phi = \frac{\partial T}{\partial\dot\phi} = I_3\cos(\theta)\left(\dot\phi\cos(\theta)+\dot\psi\right)+I_1\,\dot\phi\sin^2(\theta) = p_\psi\cos(\theta)+I_1\,\dot\phi\,\sin^2(\theta) \tag{4}$$

Recall that $p_\psi = L_3$ (i.e., the angular momentum in the body-fixed z direction) and $p_\phi = L_z$ (i.e., the angular momentum in the inertial frame z direction). Using these (constant) momenta we can rewrite the total energy much as in a Hamiltonian (but we will leave $\dot\theta$ alone):

$$E = \frac{1}{2}\,I_1\,\dot\theta^2+\frac{\left(p_\phi-p_\psi\cos(\theta)\right)^2}{2I_1\sin^2(\theta)}+mgR\cos(\theta)+\frac{p_\psi^2}{2I_3} \tag{5}$$

We now proceed to get to a less dimensioned form: we set the constants $p_\psi = I_1 a$, $p_\phi = I_1 b$, $mgR/I_1 = c^2$ where a, b, c all have units of angular frequency. We can then pull I_1 out of the energy:

$$E' = \frac{E}{I_1} = \frac{1}{2}\,\dot\theta^2+\frac{\left(b-a\cos(\theta)\right)^2}{2\sin^2(\theta)}+c^2\cos(\theta)+\frac{I_1a^2}{2I_3} = \frac{1}{2}\,\dot\theta^2+V(\theta)+\text{constant} \tag{6}$$

This expression now just involves constants and θ and $\dot\theta$; furthermore it is itself a constant. The usual logic of 1d conservation of energy applies to θ: turning points, equilibrium points, etc. In particular the minimum of $V(\theta)$ must be an equilibrium point where $\dot\theta = 0$. Working in terms of $u = \cos\theta$ note:

$$V(u) = \frac{(b-au)^2}{2(1-u^2)}+c^2u \tag{7}$$

and $V' = 0$ is a cubic equation. *Mathematica*'s results for its three roots are complex and not of much <u>immediate</u> help (i.e., not a short cut to the answer). However the usual expectations should hold: Given a value for E', θ should oscillate between two turning points.

We proceed by solving the differential equations numerically. The easiest way to get the θ equation is to differentiate E'; since E' is a constant the results must be zero:

```
e=theta'[t]^2/2 + (b-a Cos[theta[t]])^2/(2 Sin[theta[t]]^2)+
c^2 Cos[theta[t]] + I1 a^2/(2 I3)
D[%,t]
%/theta'[t]
tt=Simplify[%]
```

$$\theta'' - \cot(\theta)\left(a^2 + b^2\csc^2(\theta)\right) - a^2\cot^3(\theta) + ab\csc(\theta) + 2ab\cot^2(\theta)\csc(\theta) - c^2\sin(\theta) = 0 \tag{8}$$

The differential equation for ϕ follows from Eq. (4) above:

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \tag{9}$$

Together θ and ϕ define the direction of the body 3 axis in the inertial frame. We plot below the location of the tip of this axis. (The ψ motion is difficult to see in a spinning body. Do note that unlike free precession, $\dot{\phi}$ and $\dot{\psi}$ are not a constants.) If the spinning top is released (without a push) the initial conditions are: $\dot{\phi} = 0$, $\dot{\theta} = 0$, $\theta = \theta_0$, $b = a\cos(\theta_0)$. (The initial value of ϕ doesn't occur in the energy and can be taken to be any value, in this case zero.)

```
solution=NDSolve[Evaluate[ {tt==0,
phi'[t]==(b- a Cos[theta[t]])/Sin[theta[t]]^2, phi[0]==0,theta'[0]==0,
theta[0]==theta0} /. {b->40 Cos[1.], theta0->1, c->10, a->40}],{theta,phi},{t,0,3}]

ParametricPlot3D[
Evaluate[{Sin[theta[t]]Cos[phi[t]],Sin[theta[t]]Sin[phi[t]],Cos[theta[t]]}/. solution],
{t,0,3}]

Plot[Evaluate[ theta[t] /. solution],{t,0,3}]

Plot[Evaluate[ (b-a Cos[th])^2/(2(1-Cos[th]^2)) + c^2 Cos[th] /.
{b->40 Cos[1.], c->10, a->40}], {th,.95,1.2}]

D[(b-a u)^2/(2(1-u^2)) + c^2 u,u]
% /.  {c->10, a->40, u->Cos[1.]}
FindRoot[%,{b,40 Cos[1.]}]

Plot[Evaluate[ (b-a Cos[th])^2/(2(1-Cos[th]^2)) + c^2 Cos[th] /. {b->23.4, c->10, a->40}]
,{th,.95,1.2}]

solution=NDSolve[Evaluate[ {tt==0,
phi'[t]==(b- a Cos[theta[t]])/Sin[theta[t]]^2, phi[0]==0,theta'[0]==0,
theta[0]==theta0} /. {b->23.4, theta0->1, c->10, a->40}],{theta,phi},{t,0,3}]

ParametricPlot3D[
Evaluate[{Sin[theta[t]]Cos[phi[t]],Sin[theta[t]]Sin[phi[t]],Cos[theta[t]]}/. solution],
{t,0,3}]
```

Suggestions: keeping $b = a\cos\theta_0$ let a range from 1 to 80. Produce a non-zero initial $\dot{\phi}$ by letting $b > a\cos\theta_0$. Plot the effective potential, graphically find the expected range of θ oscillation and compare to the differential equation results. Find a b value that puts θ_0 at a minimum of the effective potential. Try non-zero values of initial $\dot{\theta}$.

Remark: Under the no-push initial conditions, the classically allowed region is defined by:

$$V(u) = \frac{(b - au)^2}{2(1 - u^2)} + c^2 u < c^2 u_0 \tag{10}$$

The roots of the resulting cubic determine the turning points.