Non-linear Pendulum Experiment

Part I: Detector

A laser shines continually on a phototransistor, except for a brief interruption when the bob swings through equilibrium (i.e., $\theta = 0$). The phototransistor circuit produces a 0.7 V signal when the beam illuminates it and a 5 V signal when the light is interrupted. Thus a positive pulse, with duration determined by the speed of the bob when it swings through equilibrium, is produced twice per period. A monostable multivibrator triggers on the rising phototransistor signal, sending its $\overline{Q}$ output, which normally remains at 5 V, to 0.7 V. The monostable's $\overline{Q}$ output remains at this low level for a period that is set by an external capacitor (and is independent of what is happening at its input from the phototransistor). Thus the monostable produces a fixed-length, negative-going pulse each time the bob goes through equilibrium. The output of the monostable is fed into a JK flip-flop, which changes its output every time it is fed a negative-going pulse edge.

Consider then what happens if the JK flip-flop's output is low just before the bob, moving to the left, goes through equilibrium. As the bob starts to interrupt the beam: the phototransistor's output starts to go to 5 V, the monostable triggers, sending the start of a negatively-going pulse to the JK flip-flop and the JK flip-flop responds to the negatively-going pulse by switching its output to high. The return trip through equilibrium results in the JK flip-flop toggling to low. As the bob returns to equilibrium, completing one period, the beam is interrupted again and the JK flip-flop switches to high. Thus the period of the JK-generated square wave is equal to the period of the pendulum.

The output of the JK flip-flop can be routed to a frequency counter to accurately record the length of the period.

Using the provided documentation, map out the detector circuit. Instead of using the long period of the pendulum, use the fan and scope to examine (and record in your manual) the output of each stage mentioned above.

Part II: Theory

Show, using energy conservation, that a physical pendulum with mass m, moment of inertia I and distance from pivot to center of mass l, satisfies the differential equation:

$$\frac{1}{2} I \dot{\theta}^2 - mgl\cos\theta = E = -mgl\cos\theta_\circ$$

where $\theta_\circ$ is the maximum angle reached during the oscillation. Using the trigonometric identity:

$$\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$$

show the period, τ, is given by

$$\tau = \tau_\circ \pi^{-1} \int_0^{\theta_\circ} \left(\sin^2\left(\frac{\theta_\circ}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right)^{-\frac{1}{2}} d\theta$$

where $\tau_\circ$ is

$$\tau_\circ = 2\pi \left(\frac{I}{mgl}\right)^{\frac{1}{2}}$$

Using the substitutions

$$z = \frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta_\circ}{2}\right)}, \qquad k = \sin\left(\frac{\theta_\circ}{2}\right)$$

show

$$\frac{\tau}{\tau_\circ} = \frac{2}{\pi} K(k)$$

where $K(k)$, called a complete elliptic integral of the first kind, is given by

$$K(k) = \int_0^1 \left(\left(1 - z^2\right)\left(1 - k^2 z^2\right)\right)^{-\frac{1}{2}} dz = \tfrac{\pi}{2}\, {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}, 1; k^2\right)$$

Marion (p. 163) reports a small angle approximation through $\theta_\circ^4$:

$$\frac{\tau}{\tau_\circ} = 1 + \frac{1}{16}\theta_\circ^2 + \frac{11}{3072}\theta_\circ^4$$

What is I for the apparatus? (Hint: it is <u>not</u> ml^2; recall the parallel axis theorem.) How does $\tau_\circ$ differ from an ideal pendulum?

Part III: Experiment

Gather the data needed to find the relationship between the period τ and the angular amplitude $\theta_\circ$. ($\theta_\circ$ may be calculated from the horizontal displacement x and l.) `SIDPLOT` τ vs. $\theta_\circ^2$ to Marion's small angle approximation and compare to the theory. `FIT` τ vs. $\theta_\circ$ to the exact functional form. How well does the theory fit the data? Find g twice using each $\tau_\circ$ (i.e., one obtained from `SIDPLOT` and one obtained from `FIT`. Make final plots of τ vs. $\theta_\circ$ and τ vs. $\theta_\circ^2$ showing error bars and the theoretical curve. Marion (p. 114, problem 3–8) reports that a cycloid pendulum is isochronous. Gather data to check this statement.

EXTRA: Use *Mathematica* to show:

$$\frac{\tau}{\tau_\circ} = 1 + \frac{1}{16}\theta_\circ^2 + \frac{11}{3072}\theta_\circ^4 + \frac{173}{737280}\theta_\circ^6 + \frac{22931}{1321205760}\theta_\circ^8 + \frac{1319183}{951268147200}\theta_\circ^{10} + \mathrm{O}(\theta_\circ{}^{12})$$