Radon's Daughters

1 Discussion

Growing up in the sophisticated suburbs of the big city we had a name for the less fortunate folks who lived in the far northern part of the state. We called them "UPers:" folks from the upper peninsula of Michigan. I can't print what they called us, but Dan Steck (1946–?) was born a UPer in Eagle River. 34 years before Dan, the other good UPer was born: Glenn T. Seaborg from Ishpeming. (I am willing to call a UPer good if either (1) they win a Nobel Prize (1951), discover more than 10 elements and get an element named after themselves (#106), or (2) they give me their hockey tickets.) Seaborg was one of Ernest Lawrence's "boys," and worked on the new elements produced in the first cyclotron. (Lawrence's "Rad Lab" in Berkeley claims nine Nobels, the first being Lawrence's for the invention of the cyclotron. Lawrence: born in Canton, South Dakota (1901), A.B. University of South Dakota (1922), M.A. UMn (1923), Ph.D. Yale (1925).)

These two UPers are connected by a letter the grade-school boy wrote to the Nobel Laureate. The boy asked how he could become a nuclear physicist, Seaborg replied and the advice worked as that's what Dan became. Dan's funding, like the funding of the early cyclotrons, comes in part from studying the health effects of radioactive substances. Like Seaborg, Dan seeks to identify substances (decay products of the naturally occurring inert gas radon) based on their characteristic decay times.

$^{222}_{86}\mathrm{Rn} \Longrightarrow {}^{218}_{84}\mathrm{Po} \Longrightarrow {}^{214}_{82}\mathrm{Pb} \longrightarrow {}^{214}_{83}\mathrm{Bi} \longrightarrow {}^{214}_{84}\mathrm{Po}, \ldots$

$^{A}_{Z}\mathrm{N1} \Longrightarrow {}^{\mathrm{A}-4}_{\mathrm{Z}-2}\mathrm{N2} + \alpha$ is α decay

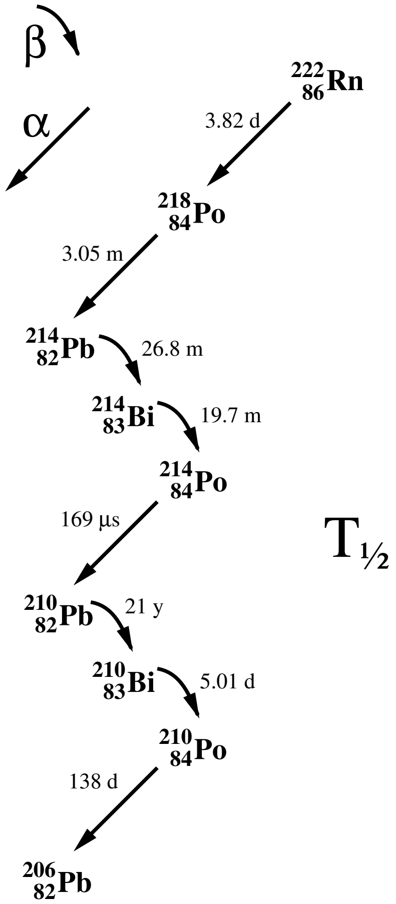
A nucleus spits out a $^{4}_{2}\mathrm{He}$ nucleus (which is the long name for an α particle).

$^{A}_{Z}\mathrm{N1} \longrightarrow {}_{\mathrm{Z}+1}^{\mathrm{A}}\mathrm{N2} + \beta + \nu$ is β decay

A nucleus spits out an electron (the electron was created from neutron (n) decay into a proton (p): $n \longrightarrow p + \beta + \nu$, or in a more modern view from the decay of a down quark inside the neutron: $d \longrightarrow u + W$ followed by: $W \longrightarrow \beta + \nu$).

The rate of these decays is wonderfully divergent: see units ranging from μs to years—a span of more than 10^{12}. The presence of radically different time-scales can be a blessing or a curse. We ignore the (chemical) mass gain as the slowly rusting iron nail falls to the ground: we say the acceleration is g. On the other hand "small" effects (like viscosity) have a way of causing important effects even through they seem negligible. Believe it or not: planes could not fly if the viscosity of air were zero instead of just "small".

Radically different scales are almost always a problem for differential equations. If we have to carefully view each micro-sec, we are never going to get to the gradual changes that accumulate over a year. (A 0.1% error compounded every micro-sec, is going to produce essentially meaningless results in less than one second, let alone one year.)



In physics we have been blessed with a variety of "disconnection" theorems. Newton tells us that the apparently simple pull of gravity on the apple is really the sum of numberless tiny pulls due to each atom that makes up the Earth. But, Newton shows, the microscopic details of the composition of the Earth have negligible effects on the apple; with symmetry we can pretend all the mass is at the center of the Earth.

We seek here a way to understand nuclear decays with out undue concern about the odd micro-sec.

Nuclear decay is a random process that has a degree of regularity.

On average, the number of decays of a radioactive substance in, say, a time dt, is proportional to the amount of material and decay constant λ:

$$dN = -\lambda N\, dt$$

Note the minus sign: there is definitely less of the substance after decay, so change in the number of nuclei dN during the time dt is negative. You all should know how to solve this differential equation:

$$N(t) = e^{-\lambda t}\, N_0$$

Note that the time after which half of the substance has decayed is given by:

$$\frac{1}{2} = e^{-\lambda T_{1/2}} \qquad \text{so: } \ln(2) = \lambda T_{1/2}$$

Thus decay constants can be expressed either in terms of $T_{1/2}$ or λ.

Things get slightly more complicated if the decaying substance produces a substance that is itself radioactive.

$$\begin{aligned} \frac{dN_1}{dt} &= -\lambda_1 N_1 \\ \frac{dN_2}{dt} &= -\lambda_2 N_2 + \lambda_1 N_1 \\ \frac{dN_3}{dt} &= -\lambda_3 N_3 + \lambda_2 N_2 \end{aligned}$$

Every decay of N_1 adds to the stock of N_2... However much N_2 decayed, that amount is added to N_3

It is useful to put all of this together into a matrix differential equation:

$$\frac{d}{dt}\begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ \vdots \end{pmatrix} = -\begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ -\lambda_1 & \lambda_2 & 0 & \cdots \\ 0 & -\lambda_2 & \lambda_3 & \cdots \\ 0 & 0 & -\lambda_3 & \cdots \end{pmatrix}\begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ \vdots \end{pmatrix}$$

Remembering that N is now a vector and M is a matrix we can write this as:

$$\frac{dN}{dt} = -M \cdot N$$

Letting our minds roam free we might say the solution is:

$$N(t) = e^{-Mt} \cdot N_0$$

and of course, if we work hard enough to define what the exponential of a matrix means, we might be right!

Clearly one way to define e^M is by Taylors series:

$$e^M \equiv 1 + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots$$

Everything is well-defined and you can show that with this definition:

$$\frac{d}{dt}e^{Mt} = Me^{Mt}$$

Having a definition nice, but it's even nicer if we can use that definition. How could we actually calculate e^M? Here there is a wonderful observation: while in general M really messes up what ever vector it operates on, for one class of objects the result is simple: eigenvectors. If $M \cdot \mathbf{a} = \alpha \mathbf{a}$ then:

$$e^M \cdot \mathbf{a} = e^\alpha \mathbf{a}$$

Lets take a look then at the first five steps of radon:

```
mat={{l1,0,0,0,0},
 {-l1,l2,0,0,0},
 {0,-l2,l3,0,0},
 {0,0,-l3,l4,0},
 {0,0,0,-l4,l5}}

{vals,vecs}=Eigensystem[mat]
```

Not surprisingly, the eigenvalues of `mat` are just the original decay rates: `{l1, l2, l3, l4, l5}` , but *note that they are not in the same order*. It is helpful to have names for the eigenvalues in the order *Mathematica* found them:

```
{a1,a2,a3,a4,a5}=vals
```

In order to calculate our solution, we must express our initial state (the vector N_0) as a linear combination of the eigenvectors:

$$N_0 = c_1 \mathbf{v_1} + \cdots + c_5 \mathbf{v_5}$$

where the c_i are constants and the $\mathbf{v_i}$ are our eigenvectors with eigenvalue α_i. At future times, the N vector is given using the time dependence of each eigenvector:

$$N(t) = c_1 e^{-\alpha_1 t}\ \mathbf{v_1} + \cdots + c_5 e^{-\alpha_5 t}\ \mathbf{v_5}$$

For example, if we initially have just N_1, so $N_0 = (N_1, 0, 0, 0, 0)$ we can express this vector as a linear combination of eigenvectors by solving the set of linear equations:

```
Solve[{c1,c2,c3,c4,c5}.vecs=={n1,0,0,0,0},{c1,c2,c3,c4,c5}]

f[t_]={c1 Exp[-a1 t],c2 Exp[-a2 t],c3 Exp[-a3 t],
       c4 Exp[-a4 t],c5 Exp[-a5 t]}.vecs /. First[%]
```

If any $T_{1/2}$ is so short that we could never observe it, we can just zero the corresponding term from the function.

2 Homework

Remember to turn in a printout showing each step as *Mathematica* solves the problem, in addition to any requested plots.

In a typical experiment Dan pumps room air through a filter for 30 minutes. ${}^{218}_{84}\mathrm{Po}$, ${}^{214}_{82}\mathrm{Pb}$ and ${}^{214}_{83}\mathrm{Bi}$ present in the air are trapped on the filter. The decays from the material collected on filter are then counted.

1. Working just with the decay sequence: ${}^{218}_{84}\mathrm{Pb} \Longrightarrow {}^{214}_{82}\mathrm{Pb} \longrightarrow {}^{214}_{83}\mathrm{Bi} \longrightarrow {}^{214}_{84}\mathrm{Po} \Longrightarrow$, find the vector-valued functions (like the above `f[t]`) which report the number of each type of nuclei as a function of time, if (A) the initial state is pure ${}^{218}_{84}\mathrm{Po}$ (call this $\mathbf{f}_{\mathrm{Po}}$), (B) the initial state is pure ${}^{214}_{82}\mathrm{Pb}$ (call this $\mathbf{f}_{\mathrm{Pb}}$), and (C) the initial state is pure ${}^{214}_{83}\mathrm{Bi}$ (you can guess what we call this). You will want to convert the given $T_{1/2}$'s to needed λ's in units of per minute.

2. Assume the room-air contains some unchanging mix of $^{218}_{84}$Po, $^{214}_{82}$Pb and $^{214}_{83}$Bi. During every minute of pumping $N_{\rm Po}$ nuclei of fresh polonium, $N_{\rm Pb}$ nuclei of fresh lead and $N_{\rm Bi}$ nuclei of fresh bismuth are captured by the filter. After 30 minutes of pumping we find a devilish mixture of undecayed $^{218}_{84}$Po, $^{214}_{82}$Pb, and $^{214}_{83}$Bi, and the products of the deposited material that did decay. The first deposited nuclei have had 30 minutes to decay before the last deposited nuclei get a chance to decay. You might think that what is needed is something like:

$$N_{\rm Po} \sum_{m=1}^{30} \mathbf{f}_{\rm Po}(t-m) + N_{\rm Pb} \sum_{m=1}^{30} \mathbf{f}_{\rm Pb}(t-m) + N_{\rm Bi} \sum_{m=1}^{30} \mathbf{f}_{\rm Bi}(t-m)$$

where $m = 1$ is the material deposited in the first minute and $m = 30$ is the material deposited in the last minute and t is the time since the start of pumping. But of course, everything is continuously deposited so what we really want is the integral:

$$N_{\rm Po} \int_{m=0}^{30} \mathbf{f}_{\rm Po}(t-m)\, dm + N_{\rm Pb} \int_{m=0}^{30} \mathbf{f}_{\rm Pb}(t-m)\, dm + N_{\rm Bi} \int_{m=0}^{30} \mathbf{f}_{\rm Bi}(t-m)\, dm$$

Have *Mathematica* do this integral.

3. Dan's counter cannot detect the β decays nor differentiate between the α decays, it just records the total α activity vs. time:

$$\lambda_1 N_1 + \lambda_4 N_4 = \texttt{\{l1,0,0,l4\}.f[t]}$$

Plot this total decay rate vs. time (`LogPlot` please, plot from when Dan has stopped pumping air ($t = 30$) to 3 hours later ($t = 210$)) if (A) $N_{\rm Po} = 10^4$, $N_{\rm Pb} = 10^4$ and $N_{\rm Bi} = 10^4$, (B) $N_{\rm Po} = 0$, $N_{\rm Pb} = 10^4$ and $N_{\rm Bi} = 10^4$, (C) $N_{\rm Po} = 10^4$, $N_{\rm Pb} = 10^3$ and $N_{\rm Bi} = 10^3$. Note that each curve has a slightly different shape (particularly immediately following pumping). Dan's job involves determining $N_{\rm Po}$, $N_{\rm Pb}$, and $N_{\rm Bi}$ from the exact shape of the experimental activity curve. Clearly you could vary the parameters $N_{\rm Po}$, $N_{\rm Pb}$, and $N_{\rm Bi}$ until the calculated curve matched the experimental curve; that is, you could perform a fit. (*Mathematica* is quite able to do such a fit, but we'll stop here.)