Digital Filters

Finite Impulse Response (non recursive)

response is for a finite time

$\rightarrow$ input is zero except at a single time

output $y_K = a_0 x_K + a_1 x_{K-1} + a_2 x_{K-2} + \cdots + a_n x_{K-n}$

eg "running average" where a's are constant $= \frac{1}{n+1}$

Note: seek response to a unit step & a unit sinusoidal at some freq ω

Infinite Impulse Response (recursive)

output $y_K = a_0 x_K + a_1 x_{K-1} + \cdots + a_n x_{K-n}$
$- (b_1 y_{K-1} + b_2 y_{K-2} + \cdots b_m y_{K-m})$

Just as a damped driven harmonic oscillator has a homogeneous solution (that decays & depends on initial conditions) and a steady particular solution that does not depend on initial conditions same applies here to digital filters — if $x_K = e^{i\omega K}$ long term $y_K = A e^{i\omega K}$ where A will depend on the frequency.

$$A\left(e^{iK\omega} + b_1 e^{i(K-1)\omega} + b_2 e^{i(K-2)\omega} + \cdots + b_m e^{i(K-m)}\right)$$

$$= a_0 e^{iK\omega} + a_1 e^{i(K-1)\omega} + a_2 e^{i(K-2)\omega} + \cdots + a_n e^{i(K-n)\omega}$$

$$\Rightarrow A = \frac{a_0 + a_1 e^{-i\omega} + \cdots + a_n e^{-in\omega}}{1 + b_1 e^{-i\omega} + b_2 e^{-2i\omega} + b_m e^{-mi\omega}}$$

so can use above to find amplitude & phase of output.

Z-transform: a way to deal with entire sequence $\{h_k\}$ at once by making it into a function

$$\mathcal{H}(z) \equiv \sum_{k=0}^{\infty} \frac{h_k}{z^k}$$

Remark: this is formal process rather than an analytical process so we're not really concerned about things like convergence, but note that the ratio test $\Rightarrow$ $\frac{(h_{k+1}/h_k)}{|z|}$ so if $\left|\frac{h_{k+1}}{h_k}\right|$ is bounded we'll have convergence for sufficiently large $|z|$. Typically $\left|\frac{h_{k+1}}{h_k}\right| \leq 1$ so we'll have convergence for $|z| > 1$. This can be a bit tricky as we will often set $z = e^{i\omega}$ where then $|z| = 1$.

Examples:

Impulse: $h_k = \{1, 0, 0, 0 \dots\}$ $\quad \mathcal{H}(z) = 1$

Step: $h_k = \{1, 1, 1, 1 \dots\}$ $\quad \mathcal{H}(z) = \frac{1}{1 - 1/z}$

decay: $h_k = \{e^{-\alpha k}\}$ $\quad \mathcal{H}(z) = \frac{1}{1 - \frac{1}{e^{\alpha} z}}$

oscillation: $h_k = \{e^{i\omega k}\}$

$$\mathcal{H}(z) = \sum \left(\frac{e^{i\omega}}{z}\right)^k = \frac{1}{1 - e^{i\omega}/z} = \frac{1}{1 - \frac{e^{i\omega}}{z}} \; \frac{1 - e^{-i\omega}/z}{1 - e^{-i\omega}/z}$$

$$= \frac{1 - e^{-i\omega}/z}{1 - \frac{2}{z}\cos\omega + \frac{1}{z^2}}$$

Note: if want Re or Im part of above, just do it.

Note: we move a h_k one step to future by multiplying by z^{-1}, Thus a 3-step running average of h_k: $g_k = \frac{1}{3}(h_k + h_{k-1} + h_{k-2})$

$$\mathcal{G}(z) = \frac{1}{3}(1 + z^{-1} + z^{-2})\,\mathcal{H}(z)$$

Examples: difference: $g_k = h_k - h_{k-1}$

$$\mathcal{G}(z) = (1 - z^{-1})\,\mathcal{H}(z)$$

inverse process: $g_k = \sum_{i=0}^{k} h_i$

$$\mathcal{G}(z) = \frac{1}{(1-z^{-1})}\,\mathcal{H}(z)$$

Remark: the above are inverses of each other — like integration & differentiation. Encryption / decryption seeks such pairs. In digital encryption (where we assume lossless-perfect-transmission) huge scrambles are possible. In Analog encryption, decryption must not be sensitive to noise.

Remark: Literature nobel prize winner Aleksandr Solzhenitsyn wrote about folks working on analog "scrambling" in Stalin's Gulags in the novel "In the First Circle", refers to Dante's view of Hell

Typical IIR Filter:

$$g_k = \sum_{i=0}^{N} a_i h_{k-i} - \sum_{i=1}^{M} b_i g_{k-i}$$

$$\mathcal{G}(z) = \underbrace{\frac{\sum a_i z^{-i}}{1 + \sum b_i z^{-i}}}_{\substack{\text{transfer function} \\ A(z)}}\,\mathcal{H}(z)$$

Remark, the transfer function is the response to an impulse: $\mathcal{H} = 1$

Remark 2: since everything is linear we can determine result of an arbitrary input as a sequence of impulses

Remark: previous work — if $x_k = e^{i\omega k}$ output (eventually) becomes $y_k = \underbrace{A(e^{i\omega})}\, e^{i\omega k}$ the magnitude of this is plotted as our Bode plot.

Example: 2 order low pass Butterworth 2000/10000

$$A(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - .3695\, z^{-1} + .1958\, z^{-2}}$$

the numerator factors: $(1 + z^{-1})^2$ so it has double zero @ $z = -1$ ie $z = e^{\pm i\pi}$ ie at Nyquist freq.

the denominator can be written $\left(1 - \frac{z_1}{z}\right)\left(1 - \frac{z_1^*}{z}\right)$

where $z_1 = .18476 + .40209\, i = .4425\, e^{i\underbrace{(1.14)}_{2\pi \frac{f}{f_0}}}$

"poles"

$f = 1814\,Hz$

Remark: the IIR of a term $\frac{1}{\left(1 - \frac{z_1}{z}\right)}$ is

$z_1^k = r^k e^{i 2\pi \frac{f}{f_0} k}$ ie a decaying amplitude at $f = 1814\,Hz$.

with a little partial fractions work

$$A(z) = 5.1068 + \underbrace{\frac{-4.1068 + 3.887\, z^{-1}}{(1 - z_1/z)(1 - z_1^*/z)}}$$

$A + B = -4.1068$

$-(A z_1^* + B z_1) = 3.887$

$$= \frac{A}{(1 - z_1/z)} + \frac{B}{(1 - z_1^*/z)}$$

$$= \frac{-2.053 - 3.890 i}{(1 - z_1/z)} + \frac{-2.053 + 3.890 i}{(1 - z_1^*/z)}$$

Example: 2 order Butterworth high pass 3000/10000

$$A(z) = \frac{1 - 2z^{-1} + z^{-2}}{1 + .3695\, z^{-1} + .1958\, z^{-2}}$$

numerator: $(1 - z^{-1})^2$ so has double zero at $z = 1$ i.e. e^{i0}

The denominator can be written $(1 - \frac{z_1}{z})(1 - \frac{z_1^*}{z})$

where $z_1 = -.18476 + .40209\, i = .4425\, e^{i\underbrace{2.0015}_{2\pi \frac{f}{f_0}}}$

$f = 3186 \leftarrow 2\pi \frac{f}{f_0}$

Partial Fraction expansion.

$$A(z) = 5.1068 - \frac{4.1068 + 3.887\, z^{-1}}{\underbrace{(1 - \frac{z_1}{z})(1 - \frac{z_1^*}{z})}_{\frac{A}{(1 - z_1/z)} + \frac{B}{(1 - z_1^*/z)}}}$$

$A + B = 4.1068$

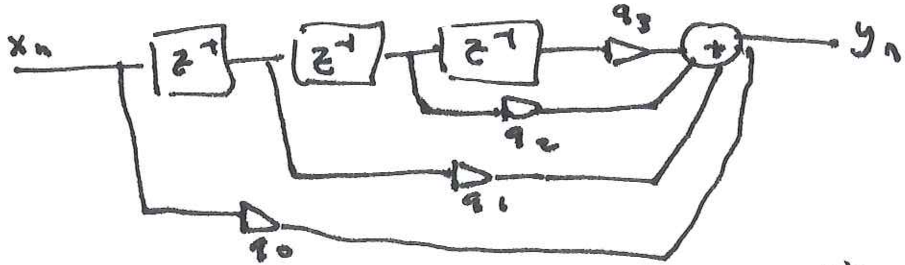
$A z_1^* + B z_1 = -3.887$

$$A(z) = 5.1068 + \frac{2.053 + 32.59\, i}{(1 - z_1/z)} + \frac{-2.053 - 32.59\, i}{(1 - z_1^*/z)}$$

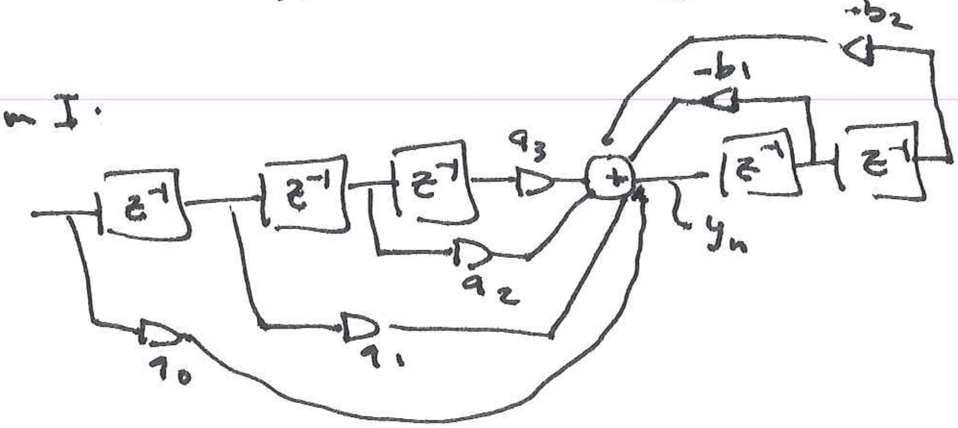
digital filter notation

$\oplus$ adder — multiplication by constant — $\boxed{z^{-1}}$ delay to previous

FIR:



IIR version I:



version II Let $A(z) = \frac{N(z)}{D(z)}$ & consider $W(z) = \frac{1}{D} X(z)$

then $Y(z) = N(z) W(z)$