T7B.1: Using 20°C=293 K as room temperature:

$$v_P = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2 \cdot 1.3807 \times 10^{-23} \cdot 293}{32 \cdot 1.6605 \times 10^{-27}}} = 390.2 \text{ m/s}$$

where I've used that the molecular mass of O_2 is 32 amu and the conversion constant is 1.6605×10^{-27} kg/amu. Now the probability, P is given by:

$$P = \frac{4}{\sqrt{\pi}} \left(\frac{v}{v_P}\right)^2 e^{-(v/v_P)^2} \frac{dv}{v_P} = \frac{4}{\sqrt{\pi}} \left(\frac{500}{390.2}\right)^2 e^{-(500/390.2)^2} \frac{10}{390.2} = 0.0184$$

T7S.1: We find max/min by finding where the derivative is zero:

$$\begin{aligned}\frac{d}{dv}\mathcal{D}(v) &= \frac{d}{dv}\left[\frac{4}{\sqrt{\pi}} \left(\frac{v}{v_P}\right)^2 e^{-(v/v_P)^2}\right] = \frac{4}{\sqrt{\pi}} \left[\left(\frac{2v}{v_P^2}\right) - \left(\frac{v}{v_P}\right)^2 \left(\frac{2v}{v_P^2}\right)\right] e^{-(v/v_P)^2} \\ &= \frac{4}{\sqrt{\pi}} \left(\frac{2v}{v_P^2}\right)\left[1 - \left(\frac{v}{v_P}\right)^2\right] e^{-(v/v_P)^2}\end{aligned}$$

This is clearly zero iff $v/v_P = 1$ and the slope is positive (negative) if $v < v_P$ ($v > v_P$)... hence a max!

`Boltzmann_Factor.problems.txt`

B. The partition function Z is:

$$Z = 1 + e^{-E_1/kT}$$

internal energy is:

$$U = N_A\langle E\rangle = N_A\left(0 + E_1 \frac{e^{-E_1/kT}}{1+e^{-E_1/kT}}\right) = \frac{N_A E_1}{e^{E_1/kT}+1} = \frac{N_A k(E_1/k)}{e^{E_1/kT}+1} = \frac{RT_\varepsilon}{e^{T_\varepsilon/T}+1}$$

where $T_\varepsilon = E_1/k = .01 \cdot 1.6022 \times 10^{-19}/(1.3807 \times 10^{-23}) = 116.04$ K. The heat capacity is:

$$c = \frac{dU}{dT} = \frac{R(T_\varepsilon/T)^2\, e^{T_\varepsilon/T}}{\left(e^{T_\varepsilon/T}+1\right)^2}$$

```
set a=116.04 b=8.3145 f(x)=b*a/(exp(a/x)+1) pmin 10 pmax 1000 xscale 2 nfun 0
fscale
border xlabel 'T (K)' ylab 'U (J)' title 'Internal Energy vs. Temperature'
fcurve
pcopy pfile 'Boltzmann_F_plotB1.eps'
clear
set f(x)=b*(a/x)^2*exp(a/x)/(exp(a/x)+1)^2
fscale
border xlabel 'T (K)' ylab 'heat capacity (J/K)' title 'Heat Capacity vs. Temperature'
fcurve
pcopy pfile 'Boltzmann_F_plotB2.eps'
```

C.
```
avgE=Sum[e n Exp[-e n/(k T)],{n,0,Infinity}]/Sum[Exp[-e n/(k T)],{n,0,Infinity}]
c=D[avgE,T]
na = 6.0221 10^23
e=.01  1.6022  10^-19
k=1.3807 10^-23
Plot[na avgE ,{T,0,300},AxesLabel ->{"T (K)","U (J)"}]
Plot[na c,{T,0,300},AxesLabel ->{"T (K)","heat capacity (J/K)"}]
Show[GraphicsArray[{{%%},{%}}]]
Export["Boltzmann_F_plotC.eps",%]
```

211t3_06.pdf

3. (a) $S_A/k_B = \ln(\Omega_A)$

   i. $U_A = 0 \Rightarrow \Omega_A = 1 \Rightarrow S_A/k_B = 0$

   ii. $U_A = 10 \Rightarrow \Omega_A = 1.88763 \times 10^{18} \Rightarrow S_A/k_B = 42.1$

   iii. $U_A = 100 \Rightarrow \Omega_A = 1.68139 \times 10^{96} \Rightarrow S_A/k_B = 222$

   iv. $U_A = 1000 \Rightarrow \Omega_A = 5.9204 \times 10^{302} \Rightarrow S_A/k_B = 697$

(b)

$$\frac{1}{T} = \frac{\partial S}{\partial U} \approx \frac{\Delta S}{\Delta U}$$

Using $\Delta S = k \ln(\Omega(U)/\Omega(U - \Delta U))$ and $\Delta U = 1\varepsilon$ obtain:

$$T = \frac{1\varepsilon}{kk \ln(\Omega(U)/\Omega(U - \Delta U))} = \frac{.005 \text{ eV}}{k} \frac{1}{\ln(\Omega(U)/\Omega(U - \Delta U))} = \frac{58.023 \text{ K}}{\ln(\Omega(U)/\Omega(U - \Delta U))}$$

   i. $U_A = 100 \Rightarrow T = 58.023 \text{ K} / \ln(1.68139/.421401) = 41.9 \text{ K}$

   ii. $U_A = 200 \Rightarrow T = 58.023 \text{ K} / \ln(3.0330/1.2156) = 63.5 \text{ K}$

   iii. $U_A = 1900 \Rightarrow T = 58.023 \text{ K} / \ln(1.2315/1.0641) = 397 \text{ K}$

   iv. $U_A = 2000 \Rightarrow T = 58.023 \text{ K} / \ln(1.9303/1.6792) = 416 \text{ K}$

(c) $100 \to 200: \quad \Delta T = 63.5 - 41.9 = 21.6 \text{ K}$

$1900 \to 2000: \quad \Delta T = 416 - 397 = 19 \text{ K}$

Note: for this problem $T_\varepsilon = \varepsilon/k = 58.023$ K, so $100 \to 200$ is for $T \approx T_\varepsilon$ whereas $1900 \to 2000$, is for much higher temperatures. Figure T7.8 shows that the specific heat of an Einstein solid at $T \approx T_\varepsilon$, is $\sim$90% of the high temperature plateau value; this is quite consistent with the $\sim$10% larger ΔT found for $100 \to 200$.