1. We are given that the function whose harmonic content we seek, $f(x)$, is either zero or one:

$$f(x) = \begin{cases} 1 & |x| < c \\ 0 & \text{elsewhere} \end{cases}$$

From class we have the expansion coefficient formula:

$$a_m = \frac{1}{\sqrt{L}} \int_0^L e^{-2\pi ixm/L} f(x) dx$$

The integrand is non-zero only for $|x| < c$ so we need only integrate there:

$$\begin{aligned} a_m &= \frac{1}{\sqrt{L}} \int_{-c}^{c} e^{-2\pi ixm/L} 1 dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-c}^{c} e^{-ixm} 1 dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{-im} \; e^{-ixm}\Big|_{-c}^{c} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{-im} \left(e^{-icm} - e^{icm}\right) \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{m} \sin(mc) \end{aligned}$$

For $c = \pi/2$, recall that $\sin(m\pi/2) = 0$ if m is even and $\sin(m\pi/2) = \pm 1$ if m is odd, so

$$a_m = \begin{cases} 0 & m \text{ even} \\ -1^{(m-1)/2} \sqrt{\frac{2}{\pi}} \frac{1}{m} & m \text{ odd} \end{cases}$$

a_0 is a special case (note the indefinite 0/0 form):

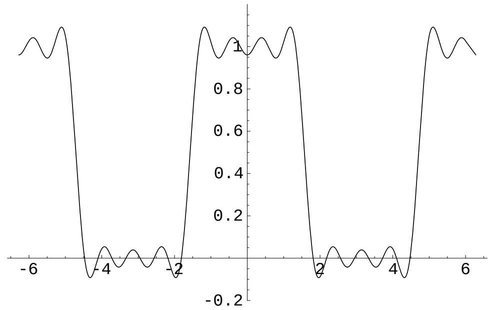
$$a_0 = \frac{1}{\sqrt{L}} \int_0^L 1 f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-c}^{c} 1\, dx = \frac{2c}{\sqrt{2\pi}}$$

In terms of the sin/cos series:

$$\begin{aligned} f(x) &= a_0 e_0(x) + \sum_{m=1}^{\infty} \mathrm{Re}(2 a_m e_m(x)) = \frac{1}{\sqrt{L}}\, a_0 + \frac{2}{\sqrt{L}} \sum_{m=1}^{\infty} \mathrm{Re}\left(a_m e^{i2\pi mx/L}\right) \\ &= \frac{1}{\sqrt{L}}\, a_0 + \frac{2}{\sqrt{L}} \left[\sum_{m=1}^{\infty} \alpha_m \cos(2\pi mx/L) - \beta_m \sin(2\pi mx/L)\right] \\ &= \frac{1}{\sqrt{L}}\, a_0 + \frac{2}{\sqrt{L}} \left[\sum_{m=1}^{\infty} A_m \cos(2\pi mx/L + \phi_m)\right] \end{aligned}$$

where we have defined the "rectangular" components of a_m by $a_m = \alpha_m + i\beta_m$ and the "polar" form of a_m by $a_m = A_m e^{i\phi_m}$. For this problem, a_m is real (i.e., $\beta_m = 0$) so we have:

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\, a_0 + \frac{2}{\sqrt{2\pi}} \left[\sum_{\text{odd}} \sqrt{\frac{2}{\pi}} \frac{-1^{(m-1)/2}}{m} \cos(mx)\right] \\ &= \frac{1}{2} + \frac{2}{\pi} \sum_{\text{odd}} \frac{-1^{(m-1)/2}}{m} \cos(mx) \end{aligned}$$



2.

$$f(x) = \begin{cases} \cos(\pi x/2c) & |x| < c \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned}
a_m &= \frac{1}{\sqrt{2\pi}} \int_{-c}^{c} e^{-ixm} \cos(\pi x/2c)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{+\pi/2} e^{-i2cym/\pi} \cos(y)\, dy\; 2c/\pi \\
&= \frac{1}{\sqrt{2\pi}} \frac{2c/\pi}{1-(2cm/\pi)^2} \left[e^{-i2cym/\pi} \left((-i2cm/\pi)\cos(y) + \sin(y) \right) \right]_{-\pi/2}^{+\pi/2} \\
&= \frac{1}{\sqrt{2\pi}} \frac{2c/\pi}{1-(2cm/\pi)^2} \left[e^{-icm} + e^{+icm} \right] \\
&= \frac{1}{\sqrt{2\pi}} \frac{4c/\pi}{1-(2cm/\pi)^2} \left[\cos(cm) \right]
\end{aligned}$$

For $c = \pi/2$, recall that $\cos(m\pi/2) = 0$ if m is odd and $\cos(m\pi/2) = \pm 1$ if m is even, so

$$a_m = \begin{cases} 0 & m \text{ odd} \\ -1^{m/2} \dfrac{1}{\sqrt{2\pi}} \dfrac{2}{1-m^2} & m \text{ even} \end{cases}$$

a_1 is a special case for $c = \pi/2$ (note the 0/0 indefinite form):

$$a_1 = \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{+\pi/2} e^{-ix} \cos(x) dx = \frac{1}{2\sqrt{2\pi}} \int_{-\pi/2}^{+\pi/2} \left(1 + e^{-2ix}\right) dx = \frac{1}{2\sqrt{2\pi}}\, \pi$$

Here $\beta_m = 0$ so for $c = \pi/2$:

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\, a_0 + \frac{1}{\sqrt{2\pi}}\, 2a_1\, \cos(x) + \frac{2}{\sqrt{2\pi}} \left[\sum_{\text{even}} \frac{2}{\sqrt{2\pi}} \frac{-1^{m/2}}{1-m^2} \cos(mx) \right] \\
&= \frac{1}{\pi} + \frac{1}{2}\, \cos(x) + \frac{2}{\pi} \sum_{\text{even}} \frac{-1^{m/2}}{1-m^2}\, \cos(mx)
\end{aligned}$$