Introduction to Complex Numbers in Physics/Engineering

Reference: Mary L. Boas, *Mathematical Methods in the Physical Sciences*
Chapter 2 & 14
George Arfken, *Mathematical Methods for Physicists*
Chapter 6

The real numbers (denoted $\mathbb{R}$) are incomplete (not closed) in the sense that standard operations applied to some real numbers do not yield a real number result (e.g., square root: $\sqrt{-1}$). It is surprisingly easy to enlarge the set of real numbers producing a set of numbers that *is* closed under standard operations: one simply needs to include $\sqrt{-1}$ (and linear combinations of it). This enlarged field of numbers, called the *complex numbers* (denoted $\mathbb{C}$), consists of numbers of the form: $z = a + b\sqrt{-1}$ where a and b are real numbers. There are lots of notations for theses numbers. In mathematics, $\sqrt{-1}$ is called i (so $z = a + bi$), whereas in electrical engineering i is frequently used for current, so $\sqrt{-1}$ is called j (so $z = a + bj$). In *Mathematica* complex numbers are constructed using `I` for i. Since complex numbers require two real numbers to specify them they can also be represented as an ordered pair: $z = (a, b)$. In any case a is called the real part of z: $a = \mathrm{Re}(z)$ and b is called the imaginary part of z: $b = \mathrm{Im}(z)$. Note that the imaginary part of any complex number is real and the imaginary part of any real number is zero. Finally there is a polar notation which reports the radius (a.k.a. absolute value or magnitude) and angle (a.k.a. phase or argument) of the complex number in the form: $r\angle\theta$. The polar notation can be converted to an algebraic expression because of a surprising relationship between the exponential function and the trigonometric functions:

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{1}$$

Thus there is a simple formula for the complex number z_1 in terms of its magnitude and angle:

$$\begin{aligned}
|z_1| &\equiv \sqrt{a^2+b^2} = r && (2)\\
a &= r\cos\theta = |z_1|\cos\theta && (3)\\
b &= r\sin\theta = |z_1|\sin\theta && (4)\\
z_1 &= a + bj = |z_1|(\cos\theta + j\sin\theta) = |z_1|e^{j\theta} && (5)
\end{aligned}$$

For example, we have the following notations for the complex number $1 + i$:

$$1 + i = 1 + j = 1 + \mathtt{I} = (1, 1) = \sqrt{2}\angle 45^\circ = \sqrt{2}e^{j\pi/4} \tag{6}$$

Note that Equation 1 can be used to express the usual trigonometric functions in terms of complex exponentials:

$$\cos\theta = \mathrm{Re}(e^{j\theta}) = \frac{e^{j\theta}+e^{-j\theta}}{2} \qquad\qquad \sin\theta = \mathrm{Im}(e^{j\theta}) = \frac{e^{j\theta}-e^{-j\theta}}{2j} \tag{7}$$

Since complex numbers are closed under the standard operations, we can define things which previously made no sense: $\log(-1)$, $\arccos(2)$, $(-1)^\pi$, $\sin(i)$, ... The complex numbers are large enough to define every function value you might want. Note that addition, subtraction,

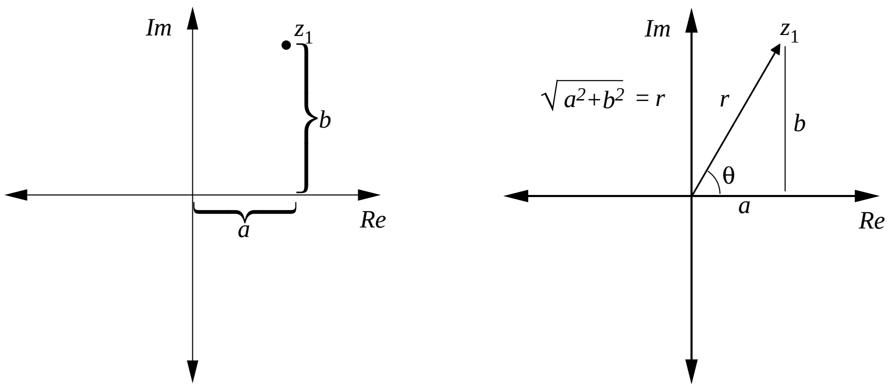


Figure 1: Complex numbers can be displayed on the complex plane. A complex number $z_1 = a + bi$ may be displayed as an ordered pair: (a, b), with the "real axis" the usual x-axis and the "imaginary axis" the usual y-axis. Complex numbers are also often displayed as vectors pointing from the origin to (a, b). The angle θ can be found from the usual trigonometric functions; $|z_1| = r$ is the length of the vector.

multiplication, and division of complex numbers proceeds as usual, just using the symbol for $\sqrt{-1}$ (let's use j):

$$z_1 = a + bj \qquad z_2 = c + dj \tag{8}$$

$$\begin{aligned}
z_1 + z_2 &= (a + bj) + (c + dj) = (a + c) + (b + d)j \\
z_1 - z_2 &= (a + bj) - (c + dj) = (a - c) + (b - d)j \\
z_1 \times z_2 &= (a + bj) \times (c + dj) = ac + adj + bcj + bdj^2 = (ac - bd) + (ad + bc)j \\
\frac{1}{z_1} &= \frac{1}{a + bj} = \frac{1}{a + bj} \times \frac{a - bj}{a - bj} = \frac{a - bj}{a^2 + b^2} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}\, j
\end{aligned}$$

Note in calculating $1/z_1$ we made use of the complex number $a - bj$; $a - bj$ is called the *complex conjugate* of z_1 and it is denoted by z_1^* or sometimes $\overline{z_1}$. See that $zz^* = |z|^2$. Note that, in terms of the ordered pair representation of $\mathbb{C}$, complex number addition and subtraction looks just like component-by-component vector addition:

$$(a, b) + (c, d) = (a + b, c + d) \tag{9}$$

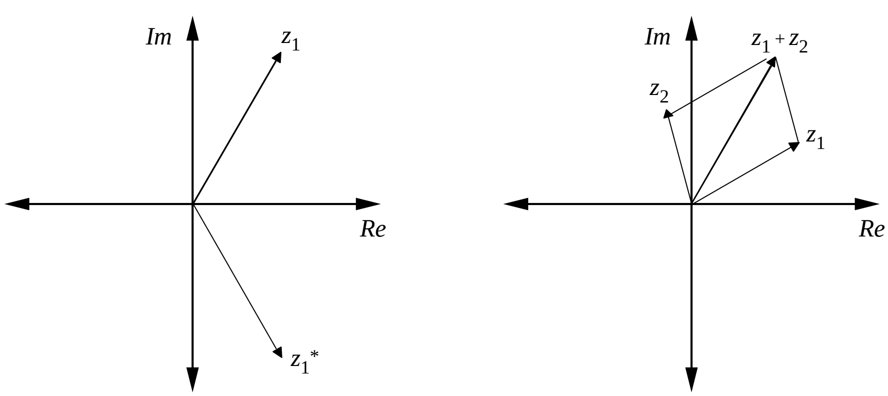


Figure 2: The complex conjugate is obtained by reflecting the vector in the real axis. Complex number addition works just like vector addition.

Thus there is a tendency to denote complex numbers as vectors rather than points in the complex plane.

Superposition of Oscillation

While the closure property of the complex numbers is dear to the hearts of mathematicians, the main use of complex numbers in science is to represent sinusoidally varying quantities in a simple way—allowing them to be combined with relative ease. (Remember that the superposition of sinusoidal quantities is itself sinusoidal, but with a new amplitude and phase.) For example, in a series RC circuit the voltage across the resistor might be given by $V_R(t) = A\cos\omega t$ whereas the voltage across the capacitor might be given by $V_C(t) = B\sin\omega t$, and the voltage across the combination (according to Kirchhoff) is the sum:

$$\begin{aligned}
V_R(t) + V_C(t) &= A\cos\omega t + B\sin\omega t \qquad \text{where: } A, B \in \mathbb{R} \\
&= \sqrt{A^2+B^2}\,\left(\frac{A}{\sqrt{A^2+B^2}}\cos\omega t + \frac{B}{\sqrt{A^2+B^2}}\sin\omega t\right) \\
&= \sqrt{A^2+B^2}\,\left(\cos\delta\cos\omega t + \sin\delta\sin\omega t\right) \qquad \text{where: } \cos\delta = \frac{A}{\sqrt{A^2+B^2}} \\
&= \sqrt{A^2+B^2}\,\cos(\omega t - \delta)
\end{aligned}$$

Yuck! That's a lot of work just to add two sinusoidal functions; we seek a simpler method (which might not seem overly simple at first glance). Note that V_R can be written as $\mathrm{Re}(Ae^{j\omega t})$ and V_C can be written as $\mathrm{Re}(-jBe^{j\omega t})$ so:

$$V_R(t) + V_C(t) = \mathrm{Re}\left((A - jB)e^{j\omega t}\right) \tag{10}$$

Now using the polar form of the complex number $A - jB$:

$$A - jB = \sqrt{A^2+B^2}\,e^{-j\phi} \qquad \text{where: } \tan\phi = B/A \tag{11}$$

we have:

$$\begin{aligned}
V_R(t) + V_C(t) &= \mathrm{Re}\left((A - jB)e^{j\omega t}\right) \\
&= \mathrm{Re}\left(\sqrt{A^2+B^2}\,e^{-j\phi}\,e^{j\omega t}\right) \\
&= \sqrt{A^2+B^2}\,\mathrm{Re}\left(e^{j(\omega t-\phi)}\right) \\
&= \sqrt{A^2+B^2}\,\cos(\omega t - \phi)
\end{aligned}$$

If we have more sinusoidal waves to add up, the problem is not much more difficult. Consider the case of adding four waves, which for ease of notation we assume have unit amplitude and constant phase difference (δ). [I'm also going to switch to $\sqrt{-1} = i$; you should get used to both notations.]

$$f(t) = \cos(\omega t) + \cos(\omega t + \delta) + \cos(\omega t + 2\delta) + \cos(\omega t + 3\delta) \tag{12}$$

We can write this as the real part of a complex expression:

$$f(t) = \mathrm{Re}\left[\left(1 + e^{i\delta} + e^{i2\delta} + e^{i3\delta}\right)\,e^{i\omega t}\right] \tag{13}$$

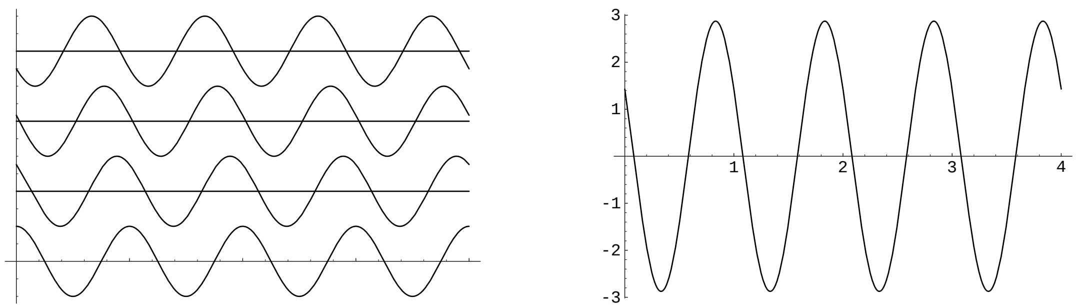


Figure 3: Consider the problem of adding four sinusoidal functions with the same frequency and amplitude, but with different offsets (see left). The result (right) is a sinusoidal function with the same frequency — we seek to easily determine the resulting amplitude and offset.

We'll use some tricks below to add these four complex numbers, but for now the main point is that they add to some complex number which can be expressed in polar form:

$$\left(1 + e^{i\delta} + e^{i2\delta} + e^{i3\delta}\right) = A\, e^{i\phi} \tag{14}$$

so

$$f(t) = \mathrm{Re}\left[A\, e^{i(\omega t+\phi)}\right] = A\cos(\omega t + \phi) \tag{15}$$

Thus adding sinusoidal waves is as simple as adding the corresponding (complex) amplitudes.

Now for the trick that applies to this particular problem. Recall the formula for the sum of the geometric series:

$$1 + r + r^2 + \cdots + r^{N-1} = \frac{1 - r^N}{1 - r} \tag{16}$$

(The proof of this result is easy: just multiply both sides by $(1 - r)$, and notice that the rhs terms telescope down to $1 - r^N$.) Here:

$$r = e^{i\delta} \tag{17}$$

So the sum is:

$$A\, e^{i\phi} = \frac{1 - e^{i4\delta}}{1 - e^{i\delta}} = \frac{e^{i2\delta}}{e^{i\delta/2}}\, \frac{e^{i2\delta} - e^{-i2\delta}}{e^{i\delta/2} - e^{-i\delta/2}} = e^{i3\delta/2}\, \frac{\sin(2\delta)}{\sin(\delta/2)} \tag{18}$$

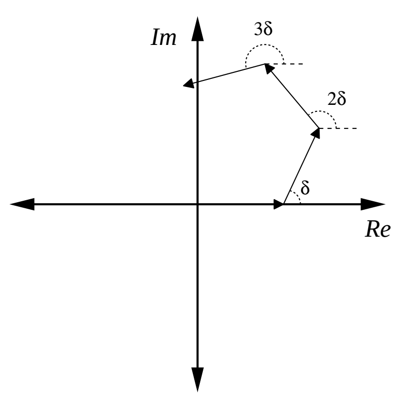


Figure 4: Finding the sum of four cosines amounts to adding four complex amplitudes: $\left(1 + e^{i\delta} + e^{i2\delta} + e^{i3\delta}\right)$

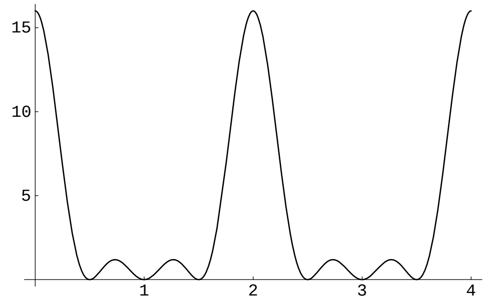


Figure 5: A^2 is plotted as a function of δ. Note that if $\delta = 0, 2\pi, 4\pi, \ldots$, the four waves are in phase so the amplitude is 4 (so $A^2 = 16$). The first zero of A^2 occurs when the four amplitude vectors form a (closed) square for $\delta = \pi/2$.

and hence:

$$A = \frac{\sin(2\delta)}{\sin(\delta/2)} \qquad\qquad \phi = 3\delta/2 \tag{19}$$

In physics we are usually most interested in the value of A^2 which is plotted in Figure 5 as function of δ.

In a more general case we might need to add N cosines with possibly different real amplitudes a_k and offsets δ_k:

$$\begin{aligned}
h(t) &= a_1\cos(\omega t+\delta_1) + a_2\cos(\omega t+\delta_2) + \cdots + a_N\cos(\omega t+\delta_N) && (20)\\
&= \sum_{k=1}^{N} a_k \cos(\omega t+\delta_k) && (21)\\
&= \mathrm{Re}\left[\left(a_1e^{i\delta_1} + a_2e^{i\delta_1} + \cdots + a_Ne^{i\delta_N}\right)\ e^{i\omega t}\right] && (22)\\
&= \mathrm{Re}\left[\left(\sum_{k=1}^{N} a_k e^{i\delta_k}\right)\ e^{i\omega t}\right] && (23)
\end{aligned}$$

Once again all that is required is to express the sum of the complex amplitudes in polar fashion:

$$\left(\sum_{k=1}^{N} a_k e^{i\delta_k}\right) = Ae^{i\phi} \tag{24}$$

then

$$h(t) = A\cos(\omega t+\phi) \tag{25}$$

Differential Equations and $e^{i\omega t}$

Complex exponentials provide a fast and easy solution for many differential equations. Consider the damped harmonic oscillator:

$$F_{\text{net}} = -kx - bv = ma \tag{26}$$

or:

$$0 = \frac{d^2x}{dt^2} + \frac{b}{m}\,\frac{dx}{dt} + \frac{k}{m}\,x \tag{27}$$

Seeking a more compact notation, we redefine the constants in this expression:

$$\frac{b}{m} \equiv 2\beta \qquad\qquad \frac{k}{m} \equiv \omega_0^2 \tag{28}$$

So:

$$\frac{d^2x}{dt^2} + 2\beta\,\frac{dx}{dt} + \omega_0^2\,x = 0 \tag{29}$$

If we guess a solution of the form: $x = Ae^{rt}$, we find:

$$\left[r^2 + 2\beta\,r + \omega_0^2\right]\ Ae^{rt} = 0 \tag{30}$$

Since e^{rt} is never zero, r must be a root of the quadratic equation in square brackets.

$$r = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega_0^2}}{2} = \beta \pm i\sqrt{\omega_0^2 - \beta^2} \tag{31}$$

where we have assumed $\omega_0 > \beta$. Defining the free oscillation frequency $\omega' = \sqrt{\omega_0^2 - \beta^2}$, we have a solution:

$$x = \mathrm{Re}\left[A\,e^{-\beta t}\ e^{i\omega' t}\right] = |A|\,e^{-\beta t}\ \cos(\omega' t + \phi) \tag{32}$$

In the driven, damped harmonic oscillator, we have a driving force: $F_0 \cos\omega t$ in addition to the other forces:

$$F_{\mathrm{net}} = F_0 \cos\omega t - kx - bv = ma \tag{33}$$

Defining $A_0 = F_0/m$ yields the differential equation:

$$\frac{d^2x}{dt^2} + 2\beta\,\frac{dx}{dt} + \omega_0^2\,x = A_0\cos(\omega t) \tag{34}$$

If we seek a solution that oscillates at the driving frequency: $x = \mathrm{Re}\left[Ae^{i\omega t}\right]$, our differential equation becomes:

$$\left[-\omega^2 + 2\beta i\omega + \omega_0^2\right] Ae^{i\omega t} = A_0 e^{i\omega t} \tag{35}$$

So:

$$A = \frac{A_0}{(\omega_0^2 - \omega^2) + 2\beta i\omega} \tag{36}$$

From which we can extract the amplitude and phase of the oscillation, for example:

$$|A| = \frac{A_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \tag{37}$$

One can show that the amplitude is largest for

$$\omega = \sqrt{\omega_0^2 - 2\beta^2} \tag{38}$$

Finally it is customary to describe driven oscillators in terms of a dimensionless *quality factor*, Q,

$$Q = \frac{\omega_0}{2\beta} \tag{39}$$

If we then let x be the dimensionless frequency ratio ω/ω_0, we can write the oscillation amplitude in a particularly simple form:

$$|A| = \frac{A_0/\omega_0^2}{\sqrt{(1 - x^2)^2 + x^2/Q^2}} \tag{40}$$

Notice that the case of small damping (small β) corresponds to large Q.

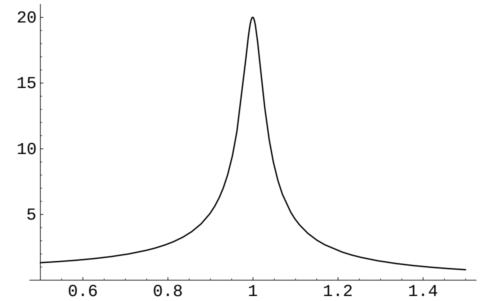
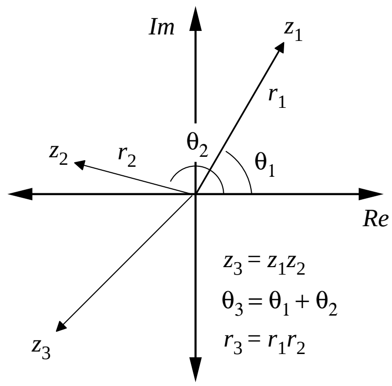


Figure 6: Resonance: the amplitude factor: $\frac{1}{\sqrt{(1-x^2)^2+x^2/Q^2}}$ is plotted as a function of the dimensionless frequency ratio: $x = \omega/\omega_0$ for the case $Q = 20$. Clearly the largest amplitude occurs when $x \approx 1$, i.e., $\omega \approx \omega_0$

Homework

1. Prove that when you multiply complex numbers z_1 and z_2, the magnitude of the result is the product of the magnitudes of z_1 and z_2, and the phase of the product is the sum of the phases of z_1 and z_2.



2. Express the following in the $r\angle\theta$ format (I bet your calculator can do this automatically):
   (a) $\dfrac{1}{1+i}$ (b) $\dfrac{3+i}{1+3i}$ (c) $25e^{2i}$ (d) $(1/(1+i))^*$ (e) $\left|\dfrac{1}{(1+i)}\right|$

3. Find the following in (a, b) format (I bet your calculator can do this automatically):
   (a) $\dfrac{3i-7}{i+4}$ (b) $(.64 + .77i)^4$ (c) $\sqrt{3+4i}$ (d) $25e^{2i}$ (e) $\ln(-1)$

4. Three cosine functions with amplitudes and offsets: $a_1 = 1.32$, $\delta_1 = .253$ rad; $a_2 = 3.21$, $\delta_2 = .532$ rad; and $a_3 = 2.13$, $\delta_3 = .325$ rad are to be added together. Find the result.

5. Find the formula for the amplitude A that results from adding 5 unit-amplitude cosine waves with constant phase difference:

$$\begin{aligned} g(t) &= \cos(\omega t) + \cos(\omega t + \delta) + \cos(\omega t + 2\delta) + \cos(\omega t + 3\delta) + \cos(\omega t + 4\delta) \\ &= A\cos(\omega t + \phi) \end{aligned}$$

   Use your favorite graphing program to make a hardcopy plot of A^2 vs. δ for $\delta \in (0, 4\pi)$.

6. Consider a driven RC circuit (see below left). According to Kirchhoff's law the voltage drop across the resistor (IR, for current I) plus the voltage drop across the capacitor (Q/C, for charge Q) must equal the voltage applied by the a.c. generator ($V_0 \cos(\omega t)$) (where the generator is operating at an angular frequency ω). Thus:

$$\frac{Q}{C} + R\,I = V_0 \cos(\omega t) \tag{41}$$

Since the the current flowing must accumulate as charge on the capacitor we have:

$$I = \frac{dQ}{dt} \tag{42}$$

Thus we have the differential equation:

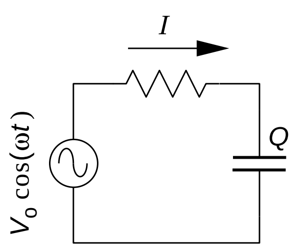
$$\frac{Q}{C} + R\,\frac{dQ}{dt} = V_0 \cos(\omega t) \tag{43}$$

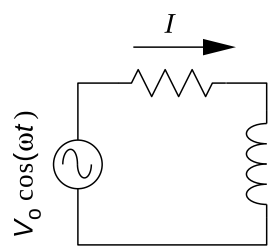
Using complex variable methods and guessing a solution of the form:

$$Q = Q_0\, e^{i\omega t} \tag{44}$$

Determine the amplitude and phase of Q_0 so you can express your final answer in real form:

$$Q = A\cos(\omega t + \phi) \tag{45}$$





7. Consider a driven RL circuit (see above right). According to Kirchhoff's law the voltage drop across the resistor (IR, for current I) plus the voltage drop across the inductor ($L\ dI/dt$) must equal the voltage applied by the a.c. generator ($V_0 \cos(\omega t)$) (where the generator is operating at an angular frequency ω). Thus:

$$L\frac{dI}{dt} + R\,I = V_0 \cos(\omega t) \tag{46}$$

Using complex variable methods and guessing a solution of the form:

$$I = I_0\, e^{i\omega t} \tag{47}$$

Determine the amplitude and phase of I_0 so you can express your final answer in real form:

$$I = A\cos(\omega t + \phi) \tag{48}$$